

ON THE FORMATION OF SHOCK FOR QUASILINEAR WAVE EQUATIONS BY PULSE WITH WEAK INTENSITY

SHUANG MIAO

ABSTRACT. In this paper we continue to study the shock formation for the 3-dimensional quasilinear wave equation

$$-(1 + 3G''(0)(\partial_t \phi)^2)\partial_t^2 \phi + \Delta \phi = 0, \quad (\star)$$

with $G''(0)$ being a non-zero constant. Since (\star) admits global-in-time solution with small initial data, to present shock formation, we consider a class of large data. Moreover, no symmetric assumption is imposed on the data. Compared to our previous work [18], here we pose data on the hypersurface $\{(t, x)|t = -r_0\}$ instead of $\{(t, x)|t = -2\}$, with r_0 being arbitrarily large. We prove an a priori energy estimate independent of r_0 . Therefore a complete description of the solution behavior as $r_0 \rightarrow \infty$ is obtained. This allows us to relax the restriction on the profile of initial data which still guarantees shock formation. Since (\star) can be viewed as a model equation for describing the propagation of electromagnetic waves in nonlinear dielectric, the result in this paper reveals the possibility to use wave pulse with weak intensity to form electromagnetic shocks in laboratory. A main new feature in the proof is that all estimates in the present paper do *not* depend on the parameter r_0 , which requires different methods to obtain energy estimates. As a byproduct, we prove the existence of semi-global-in-time solutions which lead to shock formation by showing that the limits of the initial energies exist as $r_0 \rightarrow \infty$. The proof combines the ideas in [5] where the formation of shocks for 3-dimensional relativistic compressible Euler equations with small initial data is established, and the short pulse method introduced in [6] and generalized in [15], where the formation of black holes in general relativity is proved.

1. INTRODUCTION

In this paper we study the following *quasilinear* wave equation

$$-(1 + 3G''(0)(\partial_t \phi)^2)\partial_t^2 \phi + \Delta \phi = 0, \quad (\star)$$

where $G''(0)$ is a *nonzero* constant and $\phi \in C^\infty(\mathbb{R}_t \times \mathbb{R}_x^3; \mathbb{R})$ is a smooth solution. The aim of the paper is to present an energy-estimate-based proof for the stable shock formation of smooth solutions to (\star) generated by data prescribed at $\Sigma_{-r_0} := \{(t, x)|t = -r_0\}$ with r_0 being arbitrarily large. A classical result by Klainerman [14] says that the equation (\star) admits global-in-time smooth solution with small data, so in order to prove shock formation, we consider a class of “large” data which will be specified below. As we shall see, the equation (\star) can be regarded as a model equation for the Maxwell equations in nonlinear electromagnetic theory, in which the shocks can be observed experimentally. The shock formation in nonlinear electromagnetic theory will be the subject of our forthcoming work.

1.1. Lagrangian formulation of the main equation and its relation to nonlinear electromagnetic waves. We briefly discuss the derivation of the main equation (\star) . The linear wave equation in Minkowski spacetime $(\mathbb{R}^{3+1}, m_{\mu\nu})$ can be derived by a variational principle: we take the Lagrangian density $L(\phi)$ to be $\frac{1}{2}(-(\partial_t \phi)^2 + |\nabla_x \phi|^2)$ and take the action functional $\mathcal{L}(\phi)$ to be $\int_{\mathbb{R}^{3+1}} L(\phi) d\mu_m$ where $d\mu_m$ denotes the volume form of the standard Minkowski metric $m_{\mu\nu}$. The corresponding Euler-Lagrange equation is exactly the linear wave equation $-\partial_t^2 \phi + \Delta \phi = 0$. We observe that the quadratic nature of the Lagrangian density result in the

Department of Mathematics, University of Michigan, Ann Arbor MI, U.S.A. shmiao@umich.edu
Current address: Bâtiment des Mathématiques, EPFL, Lausanne, Switzerland, shuang.miao@epfl.ch.

linearity of the equation. This simple observation allows one to derive plenty of *nonlinear* wave equations by changing the quadratic nature of the Lagrangian density. In particular, we will change the quadratic term in $\partial_t \phi$ to a quartic term, this will lead to a *quasi-linear* wave equation.

In fact, we consider a perturbation of the Lagrangian density of linear waves:

$$L(\phi) = -\frac{1}{2}G((\partial_t \phi)^2) + \frac{1}{2}|\nabla \phi|^2, \quad (1.1)$$

where $G = G(\rho)$ is a smooth function defined on \mathbb{R} and $\rho = |\partial_t \phi|^2$. The corresponding Euler-Lagrange equation is

$$-\partial_t(G'(\rho)\partial_t \phi) + \Delta \phi = 0.$$

The function $G(\rho)$ is a perturbation of $G_0(\rho) = \rho$ and therefore we can think of the above equation as a perturbation of the linear wave equation. For instance, we can work with a real analytic function $G(\rho)$ with $G(0) = 0$ and $G'(0) = 1$. In particular, we can perturb $G(\rho) = \rho$ in the simplest possible way by adding a quadratic function so that $G(\rho) = \rho + \frac{1}{2}G''(0)\rho^2$. In this situation, we obtain precisely the main equation (\star) . It is in this sense that (\star) can be regarded as the simplest quasi-linear wave equation derived from action principle.

The main equation (\star) is also closely tight to electromagnetic waves in a nonlinear dielectric. The Maxwell equations in a homogeneous insulator is derived from a Lagrangian L which is a function of the electric field E and the magnetic field B . The corresponding displacements D and H are defined through L by $D = -\frac{\partial L}{\partial E}$ and $H = \frac{\partial L}{\partial B}$ respectively. In the case of an isotropic dielectric, L is of the form

$$L = -\frac{1}{2}G(|E|^2) + \frac{1}{2}|B|^2, \quad (1.2)$$

hence $H = B$. The fields E and B are derived from the scalar potential ψ and the vector potential A according to $E = -\nabla \psi - \partial_t A$ and $B = \nabla \times A$ respectively. This is equivalent to the first pair of Maxwell equations:

$$\nabla \times E + \partial_t B = 0, \quad \nabla \cdot B = 0. \quad (1.3)$$

The potentials are determined only up to a gauge transformation $\psi \mapsto \psi - \partial_t f$ and $A \mapsto A + df$, where f is an arbitrary smooth function. The second pair of Maxwell equations

$$\nabla \cdot D = 0, \quad \nabla \times H - \partial_t D = 0. \quad (1.4)$$

are the Euler-Lagrange equations, the first resulting from the variation of ψ and the second resulting from the variation of A . Fixing the gauge by setting $\psi = 0$, we obtain a simplified model if we neglect the vector character of A replacing it by a scalar function ϕ . Then the above equations for the fields in terms of the potentials simplify to $E = -\partial_t \phi$ and $B = \nabla \phi$. The Lagrangian (1.2) becomes

$$L = -\frac{1}{2}G((\partial_t \phi)^2) + \frac{1}{2}|\nabla \phi|^2$$

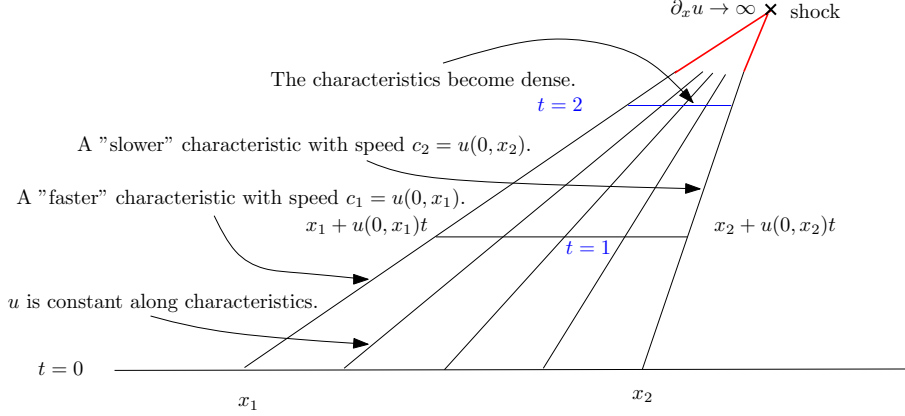
which is exactly (1.1). Therefore, the main equation (\star) provides a good approximation for shock formation in a natural physical model: the shock formations for nonlinear electromagnetic waves. We will discuss the physical motivation to start with a weak intensity pulse in detail after we introduce the initial data in Section 1.3.

1.2. A geometric perspective for shock formation. To present a geometric picture of shock formation, we start with the inviscid Burgers Equation

$$\partial_t u + u \partial_x u = 0. \quad (1.5)$$

We assume that $u \in C^\infty(\mathbb{R}_t \times \mathbb{R}_x; \mathbb{R})$ is a smooth solution. Given smooth initial data $u(0, x)$ (non-zero everywhere for simplicity), along each *characteristic curve* $u(0, x)t + x$, the solution $u(t, x)$ to (1.5) remains constant. We now consider two specific characteristics passing through x_1 and x_2 ($x_1 < x_2$). If we choose datum in such a way that $u(0, x_1) > u(0, x_2) > 0$, both the characteristics travel towards the right (see the picture below). Moreover,

the characteristic on the left (noted as C_1) travels with speed $c_1 = u(0, x_1)$ and the characteristic on the right (noted as C_2) travels with speed $c_2 = u(0, x_2)$. Since C_1 travels faster than C_2 , C_1 will eventually catch up with C_2 . The collision of two characteristics causes the breakdown on the smoothness of the solution. In summary, we have a geometric perspective on shock formation: a “faster” characteristic catches up a “slower” one so that it causes a collapse of characteristics.



In reality, instead of showing that characteristics collapse, we show that $|\partial_x u|$ blows up. Instead of being naïvely a derivative, $|\partial_x u|$ have an important *geometric interpretation*. Recall that the level sets of u are exactly the characteristic curves and the (t, x) -plane is foliated by these curves (see the above picture). Therefore, $|\partial_x u|$ is the *density of the foliation by the characteristics*. As a consequence, we can regard the shock formation as the following geometric picture: *the foliation of characteristic curves becomes infinitely dense*.

Let us recall a standard way to prove the blow-up of $|\partial_x u|$. The remarkable feature of this standard proof is that in three dimensions similar phenomenon happens for the main equation (\star) . Let $\underline{L} = \partial_t + u\partial_x$ be the tangent vectorfield of the characteristic curves (for (\star) , the corresponding vectorfield are tangent vectorfield of null geodesics on the characteristic hypersurfaces). Therefore, by taking ∂_x derivatives, we obtain

$$\underline{L}\partial_x u + (\partial_x u)^2 = 0.$$

This is a Riccati equation for $\partial_x u$ and $\partial_x u$ blows up in finite time if it is negative initially. However, we would like to understand the blow-up in another way (which is tied to the shock formation for (\star)). We define the *inverse density function* $\mu = -(\partial_x u)^{-1}$, therefore, along each characteristic curve, μ satisfies the following equation:

$$\underline{L}\mu(t, x) = -1,$$

i.e. $\underline{L}\mu$ is constant along each characteristic curve so that it is determined by its initial value. Therefore, if μ is positive initially it will eventually become 0 after finite time. $\mu \rightarrow 0$ means that the foliation becomes infinitely dense (For (\star) , we will also define an inverse density function μ for the foliation of characteristic hypersurfaces and show that $\underline{L}\mu(t, x)$ is almost a constant along each generating geodesic of the characteristic hypersurfaces).

We return to the main equation (\star) , which can be written as the following *geometric form*:

$$-\frac{1}{c^2}\partial_t^2\phi + \Delta\phi = 0 \tag{1.6}$$

with $c = (1 + 3G''(0)(\partial_t\phi)^2)^{-1/2}$. If c were a constant, (1.6) would describe the propagation of light in Minkowski spacetime. In the present situation, we still regard c as the speed of wave propagation which depends on time and position (t, x) through the unknown ϕ . We prescribe the initial data $\phi(-r_0, \theta), (\partial_t\phi)(-r_0, \theta)$ on the hypersurface Σ_{-r_0} . Let S_r be the sphere centered at the origin of Σ_{-r_0} with radius r and B_r be the ball with S_r as its boundary. We choose the data such that

- (1) The data is trivial inside B_{r_0} . By the Huygens' Principle, the solution ϕ is identically zero in the interior of the backward solid light cone with the base B_{r_0} . If we denote the wave propagation speed along the incoming characteristic hypersurface emanated from the leaf S_{r_0} by c_1 , then in view of the formula $c = (1 + 3G''(0)(\partial_t\phi)^2)^{-1/2}$, $c_1 = 1$.
- (2) $|\partial_t\phi|$ is approximately of size $\delta^{1/2}r_0^{-1}$ in the annulus region between S_{r_0} and $S_{r_0+\delta}$. If we denote the wave propagation speed along the incoming characteristic hypersurface emanated from the leaf $S_{r_0+\delta}$ by c_2 , then a Taylor expansion implies that $c_2 \sim 1 - \frac{3}{2}G''(0)\delta r_0^{-2} + O(\delta^2 r_0^{-4})$.

A key point of the paper is to identify a set of initial data such that profile $c_2(t) \sim 1 - \frac{3}{2}G''(0)\delta t^{-2} + O(\delta^2 t^{-4})$ is preserved in the evolution. Another way to understand this is through energy estimates: we can find a specific set of data so that we can obtain a priori energy estimates. Once we show that the energy (and its higher order analogue) is almost conserved, we can use Sobolev inequality to show that the pointwise profile of $\partial_t\phi$ is almost conserved along the generators of the incoming characteristic hypersurfaces. According to the formula of c , this also implies the preservation of the profile for wave speed. We are now in a situation that resembles the Burgers' picture. The initial distance between the inner most characteristic hypersurface, which is emanated from S_{r_0} , and the outer most characteristic hypersurface, which is emanated from $S_{r_0+\delta}$, is δ . We also expect the "faster"(outer) characteristic hypersurface catching up the "slower"(inner) one. This catching up process can be described as "*distance*" $= \int_{-r_0}^{time} (c_2 - c_1)dt$. Since $c_2 - c_1 \sim \delta t^{-2}$ and we regard shock formation as the collision of characteristic hypersurfaces, we expect the shock formation around $t = -1$. Finally, we point out that, as in the case of Burgers' equation, instead of showing that characteristic hypersurfaces intersect, we show that the inverse density μ of the foliation by the characteristic hypersurfaces becomes 0, i.e. the foliation turns to being infinitely dense. Similarly, this can be done by showing that profile of $\underline{L}\mu(t, x)$ is almost a constant along each generating geodesic of the characteristic hypersurfaces.

1.3. Main results. We now state the main result of the paper. Let t be the time function in Minkowski spacetime. We use Σ_t to denote its level set, which is identified as \mathbb{R}^3 for each t . We also use $\Sigma_{-r_0}^\delta$ to denote the following annulus:

$$\Sigma_{-r_0}^\delta := \{x \in \Sigma_{-r_0} \mid r_0 \leq r(x) \leq r_0 + \delta\}. \quad (1.7)$$

With the wave speed $c := (1 + 3G''(0)(\partial_t\phi)^2)^{-1/2}$, let $\underline{L} := \partial_t - c\partial_r$ and $L := \partial_t + c\partial_r$ on $\Sigma_{-r_0}^\delta$. We introduce a pair of functions $(\phi_1(s, \theta), \phi_2(s, \theta)) \in C^\infty((0, 1] \times \mathbb{S}^2)$ and we will call it the *seed data*.

The seed data $(\phi_1(s, \theta), \phi_2(s, \theta))$ can be freely prescribed and once it is given once forever. In particular, the choice of the seed data is independent of the small parameter δ . As we will see below, even though we call (ϕ_1, ϕ_2) the seed data, from the proof of the following Lemma 1.1, our initial data depends on ϕ_2 implicitly.

Lemma 1.1. *Given seed data (ϕ_1, ϕ_2) , there exists a $\delta' > 0$ depending only on the seed data, for all $\delta < \delta'$, we can construct another function $\phi_0 \in C^\infty((0, 1] \times \mathbb{S}^2)$ satisfying the following two properties:*

- (1) *For all $k \in \mathbb{Z}_{\geq 0}$, the C^k -norm of ϕ_0 are bounded by a function of the C^k -norms of ϕ_1 and ϕ_2 ;*
- (2) *If we pose initial data for (\star) on Σ_{-r_0} in the following way:*

For all $x \in \Sigma_{-r_0}$ with $r(x) \leq r_0$, we require $(\phi(-r_0, x), \partial_t\phi(-r_0, x)) = (0, 0)$; For $r_0 \leq r(x) \leq r_0 + \delta$, we require that

$$\phi(-r_0, x) = \frac{\delta^{3/2}}{r_0} \phi_0\left(\frac{r-r_0}{\delta}, \theta\right), \quad (\partial_t\phi)(-r_0, x) = \frac{\delta^{1/2}}{r_0} \phi_1\left(\frac{r-r_0}{\delta}, \theta\right). \quad (1.8)$$

Then we have

$$\|\underline{L}\phi\|_{L^\infty(\Sigma_{-r_0}^\delta)} \lesssim \frac{\delta^{3/2}}{r_0^2}, \quad \|\underline{L}^2\phi\|_{L^\infty(\Sigma_{-r_0}^\delta)} \lesssim \frac{\delta^{3/2}}{r_0^3}. \quad (1.9)$$

We remark that the condition (1.9) has a clear physical meaning: since $\underline{L}\phi$ controls the outgoing radiation, initially the waves are actually set to be incoming and the outgoing radiation is very little, because the L^∞ -norm of the outgoing radiation field $\underline{L}\phi$ is controlled by δ and its L^2 -norm on Σ_{-r_0} decays to 0 as $r_0 \rightarrow \infty$.

Proof. As in [18], a direct computation gives

$$\begin{aligned}\underline{L}^2\phi &= (2 - 3G''(0)c^3\partial_t\phi\partial_r\phi)(c^2\partial_r^2\phi) + (1 - 3G''(0)c^3\partial_t\phi\partial_r\phi)\left(\frac{2c^2}{r}\partial_r\phi + \frac{c^2}{r^2}\mathbb{A}_{\mathbb{S}^2}\phi\right) \\ &\quad + c\partial_r c\partial_r\phi - 2c\partial_r(\partial_t\phi),\end{aligned}$$

which is rewritten, in terms of ϕ_0 and ϕ_1 , as

$$\begin{aligned}\underline{L}^2\phi|_{t=-r_0} &= \left(2 - 3G''(0)c^3\phi_1\partial_s\phi_0\frac{\delta}{r_0^2}\right)\left(\frac{\delta^{-\frac{1}{2}}}{r_0}c^2\partial_s^2\phi_0\right) \\ &\quad + \left(1 - 3G''(0)c^3\phi_1\partial_s\phi_0\frac{\delta}{r_0^2}\right)\left(\frac{2c^2}{r_0^2}\partial_s\phi_0\delta^{\frac{1}{2}} + \frac{c^2}{r_0^3}\mathbb{A}_{\mathbb{S}^2}\phi_0\delta^{\frac{3}{2}}\right) \\ &\quad - 3c^4G''(0)\phi_1\partial_s\phi_1\partial_s\phi_0\frac{\delta^{\frac{1}{2}}}{r_0^3} - 2c\partial_s\phi_1\frac{\delta^{-\frac{1}{2}}}{r_0}.\end{aligned}\tag{1.10}$$

Claim: We can choose ϕ_0 , which may depend on the choice of ϕ_1 but is independent of δ , in such a way that $|\underline{L}^2\phi| \lesssim \frac{\delta^{\frac{3}{2}}}{r_0^3}$.

To see this, we first observe that since ϕ_1 is given and δ is small, we have $|c| \lesssim 1$. We make the following ansatz for ϕ_0 :

$$|\partial_s\phi_0| + |\partial_\theta^2\phi_0| \leq C,\tag{1.11}$$

where the constant C may only depend on ϕ_1 but not on δ or r_0 . By the ansatz (1.11) and by looking at the powers in δ and r_0 , one can ignore all the terms controlled by $\frac{\delta^{\frac{3}{2}}}{r_0^3}$. Therefore, to show $|\underline{L}^2\phi| \lesssim \frac{\delta^{\frac{3}{2}}}{r_0^3}$, it suffices to consider

$$\begin{aligned}&\left(2 - 3G''(0)c^3\phi_1\partial_s\phi_0\frac{\delta}{r_0^2}\right)\left(\frac{\delta^{-\frac{1}{2}}}{r_0}c^2\partial_s^2\phi_0\right) + \frac{2c^2}{r_0^2}\partial_s\phi_0\delta^{\frac{1}{2}} - 3c^4G''(0)\phi_1\partial_s\phi_1\partial_s\phi_0\cdot\frac{\delta^{\frac{1}{2}}}{r_0^3} - 2c\partial_s\phi_1\frac{\delta^{-\frac{1}{2}}}{r_0} \\ &= O\left(\frac{\delta^{\frac{3}{2}}}{r_0^3}\right),\end{aligned}$$

or equivalently,

$$\left(2 - 3G''(0)c^3\phi_1\partial_s\phi_0\frac{\delta}{r_0^2}\right)(c^2\partial_s^2\phi_0) + \frac{2c^2}{r_0}\partial_s\phi_0\delta - 3c^4G''(0)\phi_1\partial_s\phi_1\partial_s\phi_0\cdot\frac{\delta}{r_0^2} - 2c\partial_s\phi_1 = O\left(\frac{\delta^2}{r_0^2}\right).$$

Since $\left(2 - 3G''(0)c^3\phi_1\partial_s\phi_0\frac{\delta}{r_0^2}\right)^{-1} = \frac{1}{2} + \frac{3}{4}G''(0)\phi_1\partial_s\phi_0\frac{\delta}{r_0^2} + O\left(\frac{\delta^2}{r_0^4}\right)$, by multiplying both sides of the above identity by $\left(2 - 3G''(0)c^3\phi_1\partial_s\phi_0\frac{\delta}{r_0^2}\right)^{-1}$ and using the fact $c \sim 1$, it suffices to consider

$$\partial_s^2\phi_0 + \left(\frac{\delta}{r_0} + \frac{3}{2c}G''(0)\phi_1\frac{\delta}{r_0}(1 + c^2)\right)\partial_s\phi_0 = \frac{1}{c}\partial_s\phi_1 + O\left(\frac{\delta^2}{r_0^4}\right)$$

To solve for ϕ_0 , for $s \in [0, 1]$, we consider the following family (parametrized by a compact set of parameters $\theta \in \mathbb{S}^2$ and the parameters δ and $\frac{1}{r_0}$) of linear ordinary differential equations

$$\begin{aligned} \partial_s^2 \phi_0 + \left(\frac{\delta}{r_0} + \frac{3}{2c} G''(0) \phi_1 \frac{\delta}{r_0} (1+c^2) \right) \partial_s \phi_0 - c^{-1} \partial_s \phi_1 &= \frac{\delta^2}{r_0^4} \phi_2, \\ \phi_0(r_0, \theta) = \partial_s \phi_0(r_0, \theta) &= 0. \end{aligned} \quad (1.12)$$

Since the C^k -norms of the solution depends smoothly on the coefficients and the parameters θ, δ and $\frac{1}{r_0}$, all C^k -norms of ϕ_0 are of order $O(1)$ and indeed are determined by the solution of

$$\begin{aligned} \partial_s^2 \phi_0 - c^{-1} \partial_s \phi_1 &= 0, \\ \phi_0(r_0, \theta) = 0, \partial_s \phi_0(r_0, \theta) &= 0. \end{aligned} \quad (1.13)$$

In particular, this shows that the ansatz (1.11) holds if we choose C appropriately large in (1.11) and δ sufficiently small. Note that although the ODE (1.12) depends on δ and r_0^{-1} explicitly, if a priori we choose δ and r_0^{-1} to be less than $\frac{1}{2}$, the C^k -norms of ϕ_0 in (1.12) depends on ϕ_1 through (1.13) and some other absolute constants, therefore independent of the final choice of δ and r_0^{-1} . So the above construction shows that

$$|\underline{L}^2 \phi| \lesssim \frac{\delta^{\frac{3}{2}}}{r_0^3}.$$

We claim that, by the above choice of initial data, on $\Sigma_{-r_0}^\delta$, we automatically have

$$|\underline{L} \phi| \lesssim \frac{\delta^{\frac{3}{2}}}{r_0^2}.$$

Indeed, by replacing $\partial_t = \underline{L} + c\partial_r$ in the main equation, we obtain

$$\partial_r \underline{L} \phi = \frac{1}{2c} \left(-\underline{L}^2 \phi - Lc \partial_r \phi + \frac{2c^2}{r} \partial_r \phi + \frac{c^2}{r^2} \Delta_{\mathbb{S}^2} \phi \right).$$

By the construction of the data, it is obvious that all the terms on the right hand side are of size $O\left(\frac{\delta^{\frac{1}{2}}}{r_0^2}\right)$. By integrating from r_0 to r with $r \in [r_0, r_0 + \delta)$ and $\underline{L} \phi(r_0, \theta) = 0$, we have

$$|\underline{L} \phi(r, \theta)| \leq \delta \cdot O\left(\frac{\delta^{\frac{1}{2}}}{r_0^2}\right) \lesssim \frac{\delta^{\frac{3}{2}}}{r_0^2}.$$

□

Definition 1.2. *The Cauchy initial data of (\star) constructed in the lemma (satisfying the two properties) are called no-outgoing-radiation short pulse data.*

The main theorem of the paper is as follows:

Main Theorem. *For a given constant $G''(0) \neq 0$, we consider*

$$-(1 + 3G''(0)(\partial_t \phi)^2) \partial_t^2 \phi + \Delta \phi = 0.$$

Let (ϕ_1, ϕ_2) be a pair of seed data and the initial data for the equation is taken to be the no-outgoing-radiation initial data.

If the following condition on ϕ_1 holds for at least one $(s, \theta) \in (0, 1] \times \mathbb{S}^2$:

$$G''(0) \cdot \partial_s \phi_1(s, \theta) \cdot \phi_1(s, \theta) \leq -\frac{1}{3}, \quad (1.14)$$

then there exists a constant δ_0 which depends only on the seed data (ϕ_1, ϕ_2) , so that for all $\delta < \delta_0$, shocks form for the corresponding solution ϕ before $t = -1$, i.e. ϕ will no longer be smooth.

Remark 1.3. *The choice of ϕ_1 in the proof of Lemma 1.1 is arbitrary. In particular, this is **consistent** with the condition (1.14) since ϕ_1 can be freely prescribed.*

Remark 1.4. (1) *We do not assume spherical symmetry on the initial data. Therefore, the theorem is in nature a higher dimensional result.*

(2) *The proof can be applied to a large family of equations derived through action principle. We will discuss this point when we consider the Lagrangian formulation of (\star) .*

(3) *The condition (1.14) is only needed to create shocks. It is not necessary at all for the a priori energy estimates.*

Remark 1.5. *The smoothness of ϕ breaks down in the following sense:*

- 1) *The solution and its first derivative, i.e. ϕ and $\partial\phi$, are always bounded. Moreover, $|\partial_t\phi| \lesssim \delta^{\frac{1}{2}}|t|^{-1}$, therefore (\star) is always of wave type.*
- 2) *The second derivative of the solution blows up. In fact, when one approaches the shocks, $|\partial_t\partial_r\phi|$ blows up.*

1.4. A brief discussion on physical motivation. To actually create electromagnetic shocks in laboratory, one would have to focus sufficiently strong electromagnetic wave pulses into a suitable nonlinear medium. As a preliminary step for the model equation (\star) , in [18] we identified a class of large initial data, which can be thought as a strongly focused wave pulse, on the initial hypersurface $\{(t, x)|t = -2\}$, and proved shock formation before the time slice $\{(t, x)|t = -1\}$. This means that pulse has to be strong enough such that shock can form within a time period approximately 1. However, in reality due to the limitation of lasers, one can only focus weak wave pulse in experiments, namely, the initial electric field has to be small. For our model equation, $\partial_t\phi$ has to be small initially. Mathematically, this can be achieved by choosing δ sufficiently small. However, physically, if δ , which measures the width the initial pulse, is too small, then wave pulse would have high frequency and the dispersion effect would dominate. In this case, the system (1.3)-(1.4) would be no longer accurate to model the physical situation. (See [16].) Therefore the question is whether we can make the initial pulse $(\partial_t\phi|_{t=-r_0})$ small without increasing its frequency, which is measured by δ^{-1} . Thanks to the profile (1.8) of the short pulse data, by choosing r_0 sufficiently large, this problem is addressed.

1.5. Historical works. The study of singularity formation for quasilinear wave equations dates back to the work [12], in which he obtained upper bounds for the lifespan of the rotationally symmetric solutions to the equation $-\partial_t^2\phi + \Delta\phi = \partial_t\phi\partial_t^2\phi$ (see also the survey article [13] and references therein). Later in [3] and [2], Alinhac removed the symmetric assumption and showed the solution blows up in $2d$. Moreover, he gave a precise description of the solution near the blow-up point.

A major breakthrough in understanding the shock formations in higher dimensional space is made by Christodoulou in his monograph [5]. He considers the relativistic $3d$ Euler equations for a perfect irrotational fluid with an arbitrary equation of state. Given the initial data being a small perturbation from the constant state, he obtained a complete picture of shock formation. A similar result for classical Euler equations is obtained in [8]. The approaches are based on differential geometric methods originally introduced by Christodoulou and Klainerman in their monumental proof [7] of the nonlinear stability of the Minkowski spacetime in general relativity. More recently, based on similar ideas, Holzegel, Klainerman, Speck and Wong have obtained remarkable results in understanding the stable mechanism for shock formations for certain types of quasilinear wave equations with small data in three dimensions, see their overview paper [10] and Speck's detailed proof [19]. In [5] and [8] the authors obtained sharp lower and upper bounds for the lifespan of smooth solutions associated to the given data *without* any symmetry conditions. Prior to [5] [8], most of works on shock waves in fluid are limited to the simplified case of with some symmetric assumptions, i.e. essentially the one space dimension case. As an example, we note the work [1] by Alinhac in which the singularity formation for the compressible Euler equations on \mathbb{R}^2 with rotational symmetry is studied. Let us also note a recent work [9] in which a complete description of shock formation for genuinely nonlinear one-dimensional hyperbolic system

is given. [9] generalizes an influential result [11] which states that no genuinely nonlinear strictly hyperbolic quasilinear first order system in one dimensional space has a global smooth solution for small enough initial data.

All the aforementioned works have the common feature that the initial data are small. However, as we have explained at the beginning of the paper, we need to use a special family of large data—the so called *short pulse data*—to create shock for (\star) . The short pulse data was first introduced by Christodoulou in a milestone work [6] in understanding the formation of black holes in general relativity. By identifying an open set of initial data without any symmetry assumptions (the short pulse ansatz), he showed that a trapped surface can form, even in vacuum spacetime, from completely dispersed initial configurations and by means of the focusing effect of gravitational waves. Although the data are no longer close to Minkowski data, in other words, the data are no longer small, he is still able to prove a long time existence result for these data. This establishes the first result on the long time dynamics in general relativity and paves the way for many new developments on dynamical problems related to black holes. Shortly after Christodoulou's work, Klainerman and Rodnianski extends and significantly simplifies Christodoulou's work, see [15]. From a pure PDE perspective, the data appeared in the above works are carefully chosen large profiles which can be preserved by the Einstein equations along the evolution. Later in [18] the short pulse data was applied to prove shock formation for (\star) . There based on an energy estimate, we showed that the hierarchy with respect to the small parameter δ of the short pulse data is propagated until the shock formation. Since in [18] the energy estimate is proved for a finite time interval $t \in [-2, -1]$, one only needs to track the behavior of solution with respect to δ . However, in the present work, since the time behavior is crucial of interest, one needs to propagate both the δ and t hierarchy in the energy estimate. Let us conclude this subsection by noting that recently, Speck, Holzegel, Luk and Wong studied shock formation for a class of $2d$ quasilinear wave equation with initial data being a small perturbation of a plane symmetric solution (see [20], and also its recent generalization [17] to compressible Euler equation with non-zero vorticity.). Since the size of plane symmetric solution could be large, the derivative of the solution along one certain direction is allowed to be large. On the other hand, due to the smallness of the perturbation, the derivatives along other directions are small, which allows the authors to close the energy estimate. Since plane waves do not disperse, there are no dispersive estimates in [20].

1.6. New features of the proof. Since the equation (\star) is invariant under the space and time translations, we consider its linearized equation which is a linear wave equation with respect to a Lorentzian metric defined by the solution. Like in [18], the proof of the energy estimates is based on the study of geometry for the incoming null hypersurfaces with respect to this Lorentzian metric. The shock formation is equivalent to the collapse of the foliation by these null hypersurfaces. In the energy estimates, we need to use descent scheme to eliminate the singularity raised by shock formation. (See more details in the introduction of [18].) Here we will discuss some new features related to the dispersive estimates but not appearing in [18].

- (1) A modified multiplier vectorfield. In [18] since we are interested in the behavior of solution in the time interval $t \in [-2, -1]$, no dispersive estimate is needed. Therefore to prove the energy estimate, it suffice to prove that the L^2 -norm $\|(\partial\phi)(t, \cdot)\|_{L^2(\mathbb{R}^3)}$ is bounded by initial data. To prove such an energy estimate we only need to use the standard multiplier ∂_t . (In reality, we use the analog of null vectorfields $\underline{L} := \partial_t - \partial_r, L := \partial_t + \partial_r$ in Minkowski spacetime.) While in the present work, since we solve the equation from $t = -r_0$, where r_0 can be arbitrarily large, to $t = -1$, pointwise decay estimate is needed. To this end, we prove that the weighted L^2 -norm $\|t(\partial\phi)(t, \cdot)\|_{L^2(\mathbb{R}^3)}$ is bounded by initial data. For wave equations in Minkowski spacetime, the standard way to obtain such estimates is to use the conformal Killing vectorfield $K_0 := \underline{u}^2 L + u^2 \underline{L}$ as the multiplier in proving energy estimate. Here $u := \frac{1}{2}(t - r), \underline{u} := \frac{1}{2}(t + r)$. In this paper we also choose such a vectorfield as the multiplier and of course the functions u, \underline{u} and the vectorfields L, \underline{L} are associated to the Lorentzian metric defined by the solution. In reality, since we are interested in the solution defined in a region corresponding to a

small range of \underline{u} , instead of $\underline{u}^2 L + u^2 \underline{L}$, we use an analogy of $u^2 \underline{L}$ as the multiplier. (See Remark 5.4 for more discussion)

- (2) The scattering data. In the proof we will show that all the constants appearing in the estimates do not depend on r_0 . In other words, as $r_0 \rightarrow \infty$, the energy estimates still hold if the initial energies are finite as $r_0 \rightarrow \infty$. We will prove that the limits of energies as $r_0 \rightarrow \infty$ indeed exist. (See Proposition 5.2) The *scattering data* is in this sense. The energy estimates imply the semi-global existence of the smooth solution from $t = -\infty$ all the way up to shock formation. On the other hand, as we have stated, since r_0 can be made as large as we wish, the initial intensity of the pulse, $|\partial_t \phi \cdot \partial_r \partial_t \phi|(-r_0, \theta)$, can be made as small as possible without shrinking the size of δ , which fits the reality.
- (3) Resolve of a logarithmic divergence in estimating geometric quantities. To obtain the L^2 -estimates for higher order derivatives, one needs to commute certain vectorfields with the linearized equation. Since the linearized wave equation is with respect to the Lorentzian metric defined by solution, the commutator vectorfields are usually no longer Killing or conformal Killing. So we need to estimate the nonlinear contributions from the deformation tensors of commutators. In particular, the highest order derivatives of these deformation tensors are the most difficult ones to estimate. Not only that we need to modify the propagation equations satisfied by them to avoid loss of regularity (This modification also appears in [5], [8], [10], [19], [18] and [20].), but also the top order derivatives of the deformation tensors result in a logarithmic divergence:

$$E(t) \lesssim E(-r_0) + \int_{-r_0}^t (-t')^{-2} \int_{-r_0}^{t'} (-t'')^{-1} E(t'') dt'' dt' \quad (1.15)$$

Here E is the L^2 -norm of highest order derivatives for ϕ . Since $-t \leq -t' \leq -t'' \leq r_0$, the estimate for $E(t)$ would grow in r_0 . In fact, let us denote by $X(t)$ the L^2 -norm of the top order derivative of the deformation tensor. Since the contribution of $X(-r_0)$ is lower order, we omit it in this rough outline. Systematically, we have

$$X'(t) \leq (-t)^{-1} E(t) \quad \Rightarrow \quad X(t) \leq X(-r_0) + \int_{-r_0}^t (-t')^{-1} E(t') dt'. \quad (1.16)$$

On the other hand, $E(t)$ depends on $X(t)$ (roughly) through the following inequality:

$$E(t) \leq E(-r_0) + \int_{-r_0}^t (-t')^{-2} X(t') dt'. \quad (1.17)$$

Combining (1.16) and (1.17), we obtain (1.15). To avoid this logarithmic divergence, we use a modified energy $\tilde{X}(t)$ for the top order derivative of deformation tensor, such that $\tilde{X}(t) \sim (-t)^{-1} X(t)$ and $\tilde{X}'(t) \sim (-t)^{-1} X'(t)$. This modification can be achieved by modifying a commutator vectorfield. Therefore the second inequality in (1.16) becomes

$$\tilde{X}(t) \leq \tilde{X}(-r_0) + \int_{-r_0}^t (-t')^{-2} E(t') dt', \quad (1.18)$$

and the inequality in (1.17) becomes

$$E(t) \leq E(-r_0) + \int_{-r_0}^t (-t')^{-1} \tilde{X}(t') dt'. \quad (1.19)$$

Combing (1.18) and (1.19), we obtain

$$E(t) \lesssim E(-r_0) + \int_{-r_0}^t (-t')^{-1} \int_{-r_0}^{t'} (-t'')^{-2} E(t'') dt'' dt', \quad (1.20)$$

and the estimate for $E(t)$ will no longer depend on r_0 . The rigorous and detailed derivation of this argument is given in Section 7.

2. THE OPTICAL GEOMETRY

The construction of optical geometry is similar to that in [18]. To let the present paper be self-contained, here we repeat some necessary discussions in [18].

2.1. The optical metric and linearized equation. We observe that main equation (\star) is invariant under space translations, rotations and the time translation. This can also be seen from the invariance of the Lagrangian $L(\phi)$ under these symmetries. We use A to denote any possible choice from $\{\partial_t, \partial_i, \Omega_{ij} = x^i \partial_j - x^j \partial_i\}$ where $i, j = 1, 2, 3$ and $i < j$. These vectorfields correspond to the infinitesimal generators of the symmetries of (\star) .

To linearize (\star) , we apply the symmetry generated by A to a solution ϕ of (\star) to obtain a family of solutions $\{\phi_\tau : \tau \in \mathbb{R} | \phi_0 = \phi\}$. Therefore, $-\frac{1}{c(\phi_\tau)^2} \partial_t^2 \phi_\tau + \Delta \phi_\tau = 0$ for $\tau \in \mathbb{R}$. We then differentiate in τ and evaluate at $\tau = 0$. We define the so called *variations* ψ as

$$\psi := A\phi = \frac{d\phi_\tau}{d\tau} \Big|_{\tau=0} \quad (2.1)$$

By regarding ϕ as a fixed function, this procedure produces a linear equation for ψ , which is *the linearized equation of (\star) for the solution ϕ with respect to the symmetry A* .

In the tangent space at each point in \mathbb{R}^{3+1} where the solution ϕ is defined, we introduce the following Lorentzian metric $g_{\mu\nu}$

$$g = -c^2 dt \otimes dt + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3, \quad (2.2)$$

with (t, x^1, x^2, x^3) being the standard rectangular coordinates in Minkowski spacetime. Since c depends on the solution ϕ , so does $g_{\mu\nu}$. We also introduce a conformal metric $\tilde{g}_{\mu\nu}$ with the conformal factor $\Omega = \frac{1}{c}$

$$\tilde{g}_{\mu\nu} = \Omega \cdot g_{\mu\nu} = \frac{1}{c} g_{\mu\nu}. \quad (2.3)$$

We refer $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ as the *optical metric* and the *conformal optical metric* respectively.

A direct computation shows

Lemma 2.1. *The linearized equation of (\star) for a solution ϕ with respect to A can be written as*

$$\square_{\tilde{g}} \psi = 0, \quad (2.4)$$

where $\square_{\tilde{g}}$ is the wave operator with respect to \tilde{g} and $\psi = A\phi$.

This lemma can also be proved using a more natural way which is standard in Lagrangian field theory, e.g. see [4], by showing the linearized Lagrangian density of (1.1) is exactly the Lagrangian density for the linear wave equation associated to \tilde{g} .

2.2. Lorentzian geometry of the maximal development.

2.2.1. *The maximal development.* We define a function \underline{u} on Σ_{-r_0} as follows:

$$\underline{u} := r - r_0. \quad (2.5)$$

The level sets of \underline{u} in Σ_{-r_0} are denoted by $S_{-r_0, \underline{u}}$ and they are round spheres of radii $\underline{u} + r_0$. The annular region $\Sigma_{-r_0}^\delta$ defined in (1.7) is foliated by $S_{-r_0, \underline{u}}$ as

$$\Sigma_{-r_0}^\delta := \bigcup_{\underline{u} \in [0, \delta]} S_{-r_0, \underline{u}}. \quad (2.6)$$

Given an initial data set $(\phi, \partial_t \phi)|_{t=-r_0}$ defined on $B_{-r_0}^{r_0+\delta} := \bigcup_{\underline{u} \in [-r_0, \delta]} S_{-r_0, \underline{u}}$ to the main equation (\star) (as we stated in the Main Theorem), we recall the notion of *the maximal development* or maximal solution with respect to the given data.

By virtue of the local existence theorem (to (\star) with smooth data), one claims the existence of a *development* of the given initial data set, namely, the existence of

- a domain \mathcal{D} in Minkowski spacetime, whose past boundary is $B_{-r_0}^{r_0+\delta}$;
- a smooth solution ϕ to (\star) defined on \mathcal{D} with the given data on $B_{-r_0}^{r_0+\delta}$ with following property: For any point $p \in \mathcal{D}$, if an inextendible curve $\gamma : [0, \tau) \rightarrow \mathcal{D}$ satisfies the property that
 - (1) $\gamma(0) = p$,
 - (2) For any $\tau' \in [0, \tau)$, the tangent vector $\gamma'(\tau')$ is past-pointed and causal (i.e., $g(\gamma'(\tau'), \gamma'(\tau')) \leq 0$) with respect to the optical metric $g_{\alpha\beta}$ at the point $\gamma(\tau')$,
 then the curve γ must terminate at a point of $B_{-r_0}^{r_0+\delta}$.

By the standard terminology of Lorentzian geometry, the above simply says that $B_{-r_0}^{r_0+\delta}$ is a Cauchy hypersurface of \mathcal{D} .

The local uniqueness theorem asserts that if (\mathcal{D}_1, ϕ_1) and (\mathcal{D}_2, ϕ_2) are two developments of the same initial data sets, then $\phi_1 = \phi_2$ in $\mathcal{D}_1 \cap \mathcal{D}_2$. Therefore the union of all developments of a given initial data set is itself a development. This is the so called *maximal development* and its corresponding domain is denoted by W^* . The corresponding solution is called the *maximal solution*. Sometimes we also identify the development as its corresponding domain when there is no confusion.

2.2.2. *Geometric set-up.* Given an initial data set, we consider a specific family of incoming null hypersurfaces (with respect to the optical metric g) in the maximal development W^* . Recall that \underline{u} is defined on Σ_{-r_0} as $r - r_0$. For any $\underline{u} \in [0, \delta]$, we use $\underline{C}_{\underline{u}}$ to denote the incoming null hypersurface emanated from the sphere $S_{-r_0, \underline{u}}$. By definition, we have $\underline{C}_{\underline{u}} \subset W^*$ and $\underline{C}_{\underline{u}} \cap \Sigma_{-r_0} = S_{-r_0, \underline{u}}$.

Let W_δ be the subset of the maximal development of the given initial data foliated by $\underline{C}_{\underline{u}}$ with $\underline{u} \in [0, \delta]$, i.e.,

$$W_\delta := \bigcup_{\underline{u} \in [0, \delta]} \underline{C}_{\underline{u}}. \quad (2.7)$$

Roughly speaking, our main estimates will be carried out only on W_δ . The reason is as follows: since we assume that the data set is completely trivial for $\underline{u} \leq 0$ on Σ_{-r_0} , the uniqueness of smooth solutions for quasilinear wave equations implies that the spacetime in the interior of \underline{C}_0 is indeed determined by the trivial solution. In particular, \underline{C}_0 is a flat cone in Minkowski spacetime (with respect to the Minkowski metric).

We extend the function \underline{u} to W_δ by requiring that the hypersurfaces $\underline{C}_{\underline{u}}$ are precisely the level sets of the function \underline{u} . Since $\underline{C}_{\underline{u}}$ is null with respect to $g_{\alpha\beta}$, the function \underline{u} is then a solution to the equation

$$(g^{-1})^{\alpha\beta} \partial_\alpha \underline{u} \partial_\beta \underline{u} = 0, \quad (2.8)$$

where $(g^{-1})^{\alpha\beta}$ is the inverse of the metric $g_{\alpha\beta}$. We call such a function \underline{u} an *optical function*.

With respect to the affine parameter, the future-directed tangent vectorfield of a null geodesic on $\underline{C}_{\underline{u}}$ is given by

$$\widehat{\underline{L}} := -(g^{-1})^{\alpha\beta} \partial_{\alpha} \underline{u} \partial_{\beta}. \quad (2.9)$$

However, for an apparent reason, which will be seen later, instead of using $\widehat{\underline{L}}$, we will work with a renormalized (by the time function t) vectorfield \underline{L} defined through

$$\underline{L} = \mu \widehat{\underline{L}}, \quad \underline{L}t = 1, \quad (2.10)$$

i.e., \underline{L} is the tangent vectorfield of null geodesics parametrized by t .

The function μ can be computed as

$$\frac{1}{\mu} = -(g^{-1})^{\alpha\beta} \partial_{\alpha} \underline{u} \partial_{\beta} t.$$

We will see later on that the μ also has a very important geometric meaning: μ^{-1} is the density of the foliation $\bigcup_{\underline{u} \in [0, \delta]} \underline{C}_{\underline{u}}$.

Given $\underline{u} \leq \delta$, to consider the density of null-hypersurface-foliation on $\Sigma_t \cap W_{\delta}$, we define

$$\mu_m^{\underline{u}}(t) = \min \left(\inf_{(\underline{u}', \theta) \in [0, \underline{u}] \times \mathbb{S}^2} \mu(t, \underline{u}', \theta), 1 \right). \quad (2.11)$$

For $\underline{u} = \delta$, we define

$$s_* = \sup \{t \mid t \geq -r_0 \text{ and } \mu_m^{\delta}(t) > 0\}.$$

From the PDE perspective, for the given initial data to (\star) (as constructed in Lemma 1.1), we also define

$$t_* = \sup \{ \tau \mid \tau \geq -r_0 \text{ such that the smooth solution exists for all } (t, \underline{u}) \in [-r_0, \tau) \times [0, \delta] \text{ and } \theta \in \mathbb{S}^2 \}.$$

Finally, we define

$$s^* = \min\{s_*, -1\}, \quad t^* = \min\{t_*, s^*\}. \quad (2.12)$$

We remark that we will exhibit data in such a way that the solution breaks down before $t = -1$. This is the reason we take -1 in the definition of s^* .

In the sequel, we will work in a further confined spacetime domain $W_{\delta}^* \subset W_{\delta} \subset W^*$ to prove a priori energy estimates. By definition, it consists of all the points in W_{δ} with time coordinate $t \leq t^*$, i.e.,

$$W_{\delta}^* = W_{\delta} \bigcap \left(\bigcup_{-r_0 \leq t \leq t^*} \Sigma_t \right).$$

For the purpose of future use, we introduce more notations to describe various geometric objects.

For each $(t, \underline{u}) \in [-r_0, t^*) \times [0, \delta]$, we use $S_{t, \underline{u}}$ to denote the closed two dimensional surface

$$S_{t, \underline{u}} := \Sigma_t \bigcap \underline{C}_{\underline{u}}. \quad (2.13)$$

In particular, we have

$$W_{\delta}^* = \bigcup_{(t, \underline{u}) \in [-r_0, t^*) \times [0, \delta]} S_{t, \underline{u}}. \quad (2.14)$$

For each $(t, \underline{u}) \in [-r_0, t^*) \times [0, \delta]$, we define

$$\begin{aligned} \Sigma_t^{\underline{u}} &:= \{(t, \underline{u}', \theta) \in \Sigma_t \mid 0 \leq \underline{u}' \leq \underline{u}\}, \\ \underline{C}_{\underline{u}}^t &:= \{(\tau, \underline{u}, \theta) \in \underline{C}_{\underline{u}} \mid -r_0 \leq \tau \leq t\}, \\ W_{\underline{u}}^t &:= \bigcup_{(t', \underline{u}') \in [-r_0, t) \times [0, \underline{u}]} S_{t', \underline{u}'}. \end{aligned} \quad (2.15)$$

In what follows when working in $W_{\underline{u}}^t$, we usually omit the superscript \underline{u} to write $\mu_{\underline{m}}^{\underline{u}}(t)$ as $\mu_m(t)$, whenever there is no confusion.

We define the vectorfield T in W_{δ}^* by the following three conditions:

- 1) T is tangential to Σ_t ;
- 2) T is orthogonal (with respect to g) to $S_{t,\underline{u}}$ for each $\underline{u} \in [0, \delta]$;
- 3) $T\underline{u} = 1$.

The letter T stands for “transversal” since the vectorfield is transversal to the foliation of null hypersurfaces $\underline{C}_{\underline{u}}$.

In particular, the point 1) implies

$$Tt = 0. \quad (2.16)$$

According to (2.8)-(2.10), we have

$$\underline{L}u = 0, \quad \underline{L}t = 1. \quad (2.17)$$

In view of (2.10), (2.17), (2.16) and the fact $T\underline{u} = 1$, we see that the commutator

$$\Lambda := [\underline{L}, T] \quad (2.18)$$

is tangential to $S_{t,\underline{u}}$.

In view of (2.8)-(2.10) and the fact $T\underline{u} = 1$ we have

$$g(\underline{L}, T) = -\mu, \quad g(\underline{L}, \underline{L}) = 0. \quad (2.19)$$

Since T is spacelike with respect to g (indeed, Σ_t is spacelike and T is tangential to Σ_t), we denote

$$g(T, T) = \kappa^2, \quad \kappa > 0. \quad (2.20)$$

Lemma 2.2. *As in [18], we have the following relations for \underline{L} , T , μ and κ :*

$$\mu = c\kappa, \quad \underline{L} = \partial_0 - c\kappa^{-1}T, \quad (2.21)$$

where ∂_0 is the standard time vectorfield in Minkowski spacetime.

Proof. The proof is by direct computations, which can be found in [18]. We omit it here. \square

Remark 2.3. *On the initial Cauchy surface Σ_{-r_0} , since $\underline{u} = r - r_0$, we have $T = \partial_r$ and $\kappa = 1$. Therefore, by using the standard rectangular coordinates, we obtain that*

$$\underline{L} = \partial_t - c\partial_r.$$

This is coherent with the notations and computations in Lemma 1.1.

2.2.3. The optical coordinates. We construct a new coordinate system on W_{δ}^* . If shocks form, the new coordinate system is completely different from the standard rectangular coordinates. Indeed, we will show that they define two differentiable structures on W_{δ}^* when shocks form.

Given $\underline{u} \in [0, \delta]$, the generators of $\underline{C}_{\underline{u}}$ define a diffeomorphism between $S_{-r_0, \underline{u}}$ and $S_{t, \underline{u}}$ for each $t \in [-r_0, t^*)$. Since $S_{-r_0, \underline{u}}$ is diffeomorphic to the standard sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ in a natural way. We obtain a natural diffeomorphism between $S_{t, \underline{u}}$ and \mathbb{S}^2 . If local coordinates (θ^1, θ^2) are chosen on \mathbb{S}^2 , then the diffeomorphism induces local coordinates on $S_{t, \underline{u}}$ for every $(t, \underline{u}) \in [-r_0, t^*) \times [0, \delta]$. The local coordinates (θ^1, θ^2) , together with the functions (t, \underline{u}) define a complete system of local coordinates $(t, \underline{u}, \theta^1, \theta^2)$ for W_{δ}^* . This new coordinates are defined as *the optical coordinates*.

We now express for \underline{L} , T and the optical metric g in the optical coordinates.

First of all, the integral curves of \underline{L} are the lines with constant \underline{u} and θ . Since $\underline{L}t = 1$, therefore in optical coordinates we have

$$\underline{L} = \frac{\partial}{\partial t}. \quad (2.22)$$

Similarly, since $T\underline{u} = 1$ and T is tangential to Σ_t , we have

$$T = \frac{\partial}{\partial \underline{u}} - \Xi \quad (2.23)$$

with Ξ a vectorfield tangential to $S_{t,\underline{u}}$. Locally, we can express Ξ as

$$\Xi = \sum_{A=1,2} \Xi^A \frac{\partial}{\partial \theta^A}, \quad (2.24)$$

The metric g then can be written in the optical coordinates $(t, \underline{u}, \theta^1, \theta^2)$ as

$$g = -2\mu dt d\underline{u} + \kappa^2 d\underline{u}^2 + g_{AB} (d\theta^A + \Xi^A d\underline{u}) (d\theta^B + \Xi^B d\underline{u}) \quad (2.25)$$

with

$$g_{AB} = g \left(\frac{\partial}{\partial \theta^A}, \frac{\partial}{\partial \theta^B} \right), \quad 1 \leq A, B \leq 2. \quad (2.26)$$

To study the differentiable structure defined by the optical coordinates, we study the Jacobian Δ of the transformation from the optical coordinates $(t, \underline{u}, \theta^1, \theta^2)$ to the rectangular coordinates (x^0, x^1, x^2, x^3) .

First of all, since $x^0 = t$, we have

$$\frac{\partial x^0}{\partial t} = 1, \quad \frac{\partial x^0}{\partial \underline{u}} = \frac{\partial x^0}{\partial \theta^A} = 0.$$

Secondly, by (2.23), we can express $T = T^i \partial_i$ in the rectangular coordinates (x^1, x^2, x^3) as

$$T^i = \frac{\partial x^i}{\partial \underline{u}} - \sum_{A=1,2} \Xi^A \frac{\partial x^i}{\partial \theta^A}$$

In view of the fact that T is orthogonal to $\frac{\partial}{\partial \theta^A}$ with respect to the Euclidean metric (which is the induced metric of g on Σ_t !), we have

$$\Delta = \det \begin{pmatrix} T^1 & T^2 & T^3 \\ \frac{\partial x^1}{\partial \theta^1} & \frac{\partial x^2}{\partial \theta^1} & \frac{\partial x^3}{\partial \theta^1} \\ \frac{\partial x^1}{\partial \theta^2} & \frac{\partial x^2}{\partial \theta^2} & \frac{\partial x^3}{\partial \theta^2} \end{pmatrix} = \|T\| \left\| \frac{\partial}{\partial \theta^1} \wedge \frac{\partial}{\partial \theta^2} \right\| = c^{-1} \mu \sqrt{\det g},$$

where $\|\cdot\|$ measures the magnitude of a vectorfield with respect to the Euclidean metric in \mathbb{R}^3 (defined by the rectangular coordinates (x^1, x^2, x^3)).

We end the discussion by an important remark.

Remark 2.4 (Geometric meaning of μ). *In the sequel, we will show that the wave speed function c will be always approximately equal to 1 in W_δ^* . Since $\mu = c\kappa$, we may think of μ being κ in a efficient way.*

On the other hand, by the definition of T , in particular $T\underline{u} = 1$, we know that κ^{-1} is indeed the density of the foliation by the $\underline{C}_{\underline{u}}$'s. This is because $g(T, T) = \kappa^2$. Since the optical metric coincides with the Euclidean metric on each constant time slice Σ_t , by $\mu \sim \kappa$, we arrive at the following conclusion:

- μ^{-1} measures the foliation of the incoming null hypersurfaces $\underline{C}_{\underline{u}}$'s.

Therefore, by regarding shock formation as the collapsing (i.e. the density blows up) of the characteristics (\simeq the incoming null hypersurfaces), we may say that

- Shock formation is equivalent to $\mu \rightarrow 0$.

By virtue of the formula $\Delta = c^{-1}\mu\sqrt{\det g}$, it is clear (the volume element $\sqrt{\det g}$ will be controlled in the sequel) that if shock forms then the coordinate transformation between the optical coordinates and the rectangular coordinates will fail to be a diffeomorphism. Therefore, we can also say that

- Shock formation is equivalent to the fact that the optical coordinates on the maximal development defines a different differentiable structure (compared to the usual differentiable structure induced from the Minkowski spacetime).

2.3. Connection, curvature and structure equations. The 2nd fundamental forms of $S_{t,u} \hookrightarrow \underline{C}_u$ and $S_{t,u} \hookrightarrow \Sigma_t$ are defined as

$$\underline{\chi}_{AB} := g(\nabla_{X_A} \underline{L}, X_B), \quad \theta_{AB} := g(\nabla_{X_A} \hat{T}, X_B) \quad (2.27)$$

respectively. Here ∇ is the Levi-Civita connection of g and $\hat{T} := c\mu^{-1}T$. For $\underline{\chi}$, sometimes we will treat the trace/traceless parts with respect to g separately, which are defined as $\text{tr} \underline{\chi} := (\underline{g}^{-1})^{AB} \underline{\chi}_{AB}$ and $\hat{\underline{\chi}}_{AB} := \underline{\chi}_{AB} - \frac{1}{2} \underline{g}_{AB} \text{tr} \underline{\chi}$. At each point $p \in W^*$, the vectorfields $\{\underline{L}, T\}$ form a basis of subspace of $T_p W^*$, which is orthogonal to $T_p S_{t,u}$. Since by virtue of (2.2) the vectorfield $\frac{\partial}{\partial x^0}$ is orthogonal to $T_p S_{t,u}$, $\frac{\partial}{\partial x^0}$ is a linear combination of \underline{L} and T . Actually $\frac{\partial}{\partial x^0} = \underline{L} + c\hat{T}$ and $\underline{\chi} = -c\theta$.

The torsion one forms $\underline{\eta}_A$ and $\underline{\zeta}_A$ are defined by $\underline{\zeta}_A = g(\nabla_{X_A} \underline{L}, T)$ and $\underline{\eta}_A = -g(\nabla_{X_A} T, \underline{L})$. They are related to the inverse density μ by $\underline{\eta}_A = \underline{\zeta}_A + X_A(\mu)$ and $\underline{\zeta}_A = -c^{-1}\mu X_A(c)$.

An outgoing null vectorfield

$$L := c^{-2}\mu \underline{L} + 2T \quad (2.28)$$

is introduced so that $g(\underline{L}, L) = -2\mu$. The corresponding 2nd fundamental form is $\chi_{AB} = g(\nabla_{X_A} L, X_B)$. Similarly, we define $\text{tr} \chi = \text{tr}_g \chi = \underline{g}^{AB} \chi_{AB}$ and $\hat{\chi}_{AB} = \chi_{AB} - \frac{1}{2} \text{tr} \chi \underline{g}_{AB}$.

The covariant derivative ∇ is expressed in the frame $(\underline{L}, T, X_1, X_2)$:

$$\begin{aligned} \nabla_{\underline{L}} \underline{L} &= \mu^{-1}(\underline{L}\mu)\underline{L}, \quad \nabla_T \underline{L} = \underline{\eta}_A X_A - c^{-1}\underline{L}(c^{-1}\mu)\underline{L}, \\ \nabla_{X_A} \underline{L} &= -\mu^{-1}\underline{\zeta}_A \underline{L} + \underline{\chi}_A{}^B X_B, \quad \nabla_{\underline{L}} T = -\underline{\zeta}^A X_A - c^{-1}\underline{L}(c^{-1}\mu)\underline{L}, \\ \nabla_T T &= c^{-3}\mu(Tc + \underline{L}(c^{-1}\mu))\underline{L} + \left(c^{-1}(Tc + \underline{L}(c^{-1}\mu)) + T(\log(c^{-1}\mu))\right)T - c^{-1}\mu \underline{g}^{AB} X_B(c^{-1}\mu)X_A, \\ \nabla_{X_A} T &= \mu^{-1}\underline{\eta}_A T + c^{-1}\mu \theta_{AB} \underline{g}^{BC} X_C, \quad \nabla_{\underline{L}} X_A = \nabla_{X_A} \underline{L}, \quad \nabla_{X_A} X_B = \nabla_{X_A} X_B + \mu^{-1}\underline{\chi}_{AB} T. \end{aligned}$$

In terms of null frames $(L, \underline{L}, X_1, X_2)$, we have

$$\begin{aligned} \nabla_L \underline{L} &= -\underline{L}(c^{-2}\mu)\underline{L} + 2\underline{\eta}^A X_A, \quad \nabla_{\underline{L}} L = -2\underline{\zeta}^A X_A, \\ \nabla_L L &= (\mu^{-1}L\mu + \underline{L}(c^{-2}\mu))L - 2\mu X^A(c^{-2}\mu)X_A, \quad \nabla_{X_A} L = \mu^{-1}\underline{\eta}_A L + \chi_A{}^B X_B, \\ \nabla_{X_A} X_B &= \nabla_{X_A} X_B + \frac{1}{2}\mu^{-1}\underline{\chi}_{AB} L + \frac{1}{2}\mu^{-1}\chi_{AB}\underline{L}. \end{aligned}$$

Here ∇ is the induced covariant derivative on $S_{t,u}$.

In the Cartesian coordinates, the only non-vanishing curvature components are R_{0i0j} 's:

$$R_{0i0j} = \frac{1}{2} \frac{d(c^2)}{d\rho} \nabla_i \nabla_j \rho + \frac{1}{2} \frac{d^2(c^2)}{d\rho^2} \nabla_i \rho \nabla_j \rho - \frac{1}{4} c^{-2} \left| \frac{d(c^2)}{d\rho} \right|^2 \nabla_i \rho \nabla_j \rho.$$

In the optical coordinates, the only nonzero curvature components are $\alpha_{AB} = R(X_A, \underline{L}, X_B, \underline{L})$:

$$\alpha_{AB} = \frac{1}{2} \frac{d(c^2)}{d\rho} \nabla_{X_A, X_B}^2 \rho - \frac{1}{2} \mu^{-1} \frac{d(c^2)}{d\rho} T(\rho) \underline{\chi}_{AB} + \frac{1}{2} \left(\frac{d^2(c^2)}{d\rho^2} - \frac{1}{2} c^{-2} \left| \frac{d(c^2)}{d\rho} \right|^2 \right) X_A(\rho) X_B(\rho).$$

We define $\underline{\alpha}'_{AB} = \frac{1}{2} \frac{d(c^2)}{d\rho} \nabla_{X_A, X_B}^2 \rho + \frac{1}{2} [\frac{d^2(c^2)}{d\rho^2} - \frac{1}{2} c^{-2} |\frac{d(c^2)}{d\rho}|^2] X_A(\rho) X_B(\rho)$ such that

$$\underline{\alpha}_{AB} = -\frac{1}{2} \mu^{-1} \frac{d(c^2)}{d\rho} T(\rho) \underline{\chi}_{AB} + \underline{\alpha}'_{AB}. \quad (2.29)$$

Remark 2.5. As a convention, we say that the first term on the right hand side of (2.29) is singular in μ (since μ may go to zero). The second term $\underline{\alpha}'_{AB}$ is regular in μ .

Indeed, in the course of the proof, we will see that $\underline{\alpha}'_{AB}$ are bounded and $\underline{\alpha}_{AB}$ behaves exactly as μ^{-1} in amplitude. Therefore, in addition to two equivalent descriptions of the shock formation in Remark 2.4, we have another geometric interpretation:

- Shock formation is equivalent to the fact that curvature tensor of the optical metric g becomes unbounded.

Compared to the one dimensional picture of shock formations in conservation laws, e.g., for inviscid Burgers equation, this new description of shock formation is purely geometric in the following sense: it does not even depend on the choice of characteristic foliation (because the curvature tensor is tensorial!).

In the frame $(T, \underline{L}, \frac{\partial}{\partial \theta^A})$, we have the following structure equations in optical coordinates:

$$\underline{L}(\underline{\chi}_{AB}) = \mu^{-1}(\underline{L}\mu) \underline{\chi}_{AB} + \underline{\chi}_A^C \underline{\chi}_{BC} - \underline{\alpha}_{AB}, \quad (2.30)$$

$$\underline{d}\text{iv} \underline{\chi} - \underline{d}\text{tr} \underline{\chi} = -\mu^{-1}(\underline{\zeta} \cdot \underline{\chi} - \underline{\zeta} \text{tr} \underline{\chi}), \quad (2.31)$$

$$\underline{\mathcal{L}}_T \underline{\chi}_{AB} = (\nabla \hat{\otimes} \underline{\eta})_{AB} + \mu^{-1}(\underline{\zeta} \hat{\otimes} \underline{\eta})_{AB} - c^{-1} \underline{L}(c^{-1} \mu) \underline{\chi}_{AB} + c^{-1} \mu (\theta \hat{\otimes} \underline{\chi})_{AB}, \quad (2.32)$$

which will be used in proving the main estimates. Here $(\underline{\zeta} \cdot \underline{\chi})_B = \not{g}^{AC} \underline{\zeta}_A \underline{\chi}_{BC}$, $(\nabla \hat{\otimes} \underline{\eta})_{AB} = \frac{1}{2}(\nabla_A \underline{\eta}_B + \nabla_B \underline{\eta}_A)$, $(\underline{\zeta} \hat{\otimes} \underline{\eta})_{AB} = \frac{1}{2}(\underline{\zeta}_A \underline{\eta}_B + \underline{\zeta}_B \underline{\eta}_A)$ and $(\theta \hat{\otimes} \underline{\chi})_{AB} = \frac{1}{2}(\theta_{AC} \underline{\chi}_B^C + \theta_{BC} \underline{\chi}_A^C)$. By taking the trace of (2.30), we have

$$\underline{L} \text{tr} \underline{\chi} = \mu^{-1}(\underline{L}\mu) \text{tr} \underline{\chi} - |\underline{\chi}|_{\not{g}}^2 - \text{tr} \underline{\alpha}. \quad (2.33)$$

The inverse density function μ satisfies the following transport equation:

$$\underline{L}\mu = m + \mu e, \quad (2.34)$$

with $m = -\frac{1}{2} \frac{d(c^2)}{d\rho} T\rho$ and $e = \frac{1}{2c^2} \frac{d(c^2)}{d\rho} \underline{L}\rho$. With these notations, we have $\underline{\alpha}_{AB} = \mu^{-1} m \underline{\chi}_{AB} + \underline{\alpha}'_{AB}$.

Regarding the regularity in μ , we use (2.34) to replace $\underline{L}\mu$ in (2.30). This yields

$$\underline{L}(\underline{\chi}_{AB}) = e \underline{\chi}_{AB} + \underline{\chi}_A^C \underline{\chi}_{BC} - \underline{\alpha}'_{AB}. \quad (2.35)$$

Compared to the original (2.30), the new equation is *regular* μ in the sense that it has no μ^{-1} terms.

2.4. Rotation Vectorfields. Although $g|_{\Sigma_t}$ is flat, the foliation $S_{t, \underline{u}}$ is different from the standard spherical foliation. In the Cartesian coordinates on Σ_t , let $\Omega_1 = x^2 \partial_3 - x^3 \partial_2$, $\Omega_2 = x^3 \partial_1 - x^1 \partial_3$ and $\Omega_3 = x^1 \partial_2 - x^2 \partial_1$ be the standard rotations. Let Π be the orthogonal projection to $S_{t, \underline{u}}$. The rotation vectorfields $R_i \in \Gamma(TS_{t, \underline{u}})$ ($i = 1, 2, 3$) are defined by

$$R_i = \Pi \Omega_i. \quad (2.36)$$

Let indices $i, j, k \in \{1, 2, 3\}$. We use the T^k , \underline{L}^k and X_A^k to denote the components for T , \underline{L} and X_A in the Cartesian frame $\{\partial_i\}$ on Σ_t (notice that \underline{L} has also a 0th component $\underline{L}^0 = 1$). We introduce some functions to measure the difference between the foliations $S_{t, \underline{u}}$ and the standard spherical foliations.

The functions λ_i 's measure the derivation from R_i to Ω_i :

$$\lambda_i \hat{T} = \Omega_i - R_i. \quad (2.37)$$

The functions y'^k 's measure the derivation from \hat{T} to the standard radial vectorfield $\frac{x^i}{r} \partial_i$:

$$y'^k = \hat{T}^k - \frac{x^k}{r}. \quad (2.38)$$

We also define (we will show that $|y^k - y'^k|$ is bounded by a negligible small number)

$$y^k = \widehat{T}^k - \frac{x^k}{\underline{u} - t}. \quad (2.39)$$

The functions z^k 's measure the derivation of \underline{L} from $\partial_t - \partial_r$ in Minkowski spacetime:

$$z^k = \underline{L}^k + \frac{x^k}{\underline{u} - t} = -\frac{(c-1)x^k}{\underline{u} - t} - cy^k.$$

Finally, the rotation vectorfields can be expressed as

$$R_i = \Omega_i - \lambda_i \sum_{j=1}^3 \widehat{T}^j \partial_j, \quad \lambda_i = \sum_{j,k,l=1}^3 \varepsilon_{ilk} x^l y^k, \quad (2.40)$$

where ε_{ijk} is the totally skew-symmetric symbol.

3. INITIAL DATA, BOOTSTRAP ASSUMPTIONS AND THE MAIN ESTIMATES

3.1. Preliminary estimates on initial data. In the Main Theorem, we take the so called short pulse datum for (\star) on $\Sigma_{-r_0}^\delta$. Recall that $\phi(-r_0, x) = \frac{\delta^{3/2}}{r_0} \phi_0(\frac{r-r_0}{\delta}, \theta)$ and $\partial_t \phi(-r_0, x) = \frac{\delta^{1/2}}{r_0} \phi_1(\frac{r-r_0}{\delta}, \theta)$, where $\phi_0, \phi_1 \in C_0^\infty((0, 1] \times \mathbb{S}^2)$. The condition (3) in the statement of the Main Theorem reads as

$$\|\underline{L}\phi\|_{L^\infty(\Sigma_{-r_0})} \lesssim \frac{\delta^{3/2}}{r_0^2}, \quad \|\underline{L}^2\phi\|_{L^\infty(\Sigma_{-r_0})} \lesssim \frac{\delta^{3/2}}{r_0^2}.$$

We now derive estimates for ϕ and its derivatives on Σ_{-r_0} . These estimates also suggest the estimates, e.g. the bootstrap assumptions in next subsection, that one can expect later on.

For ϕ and $\psi = A\phi$ where $A \in \{\partial_\alpha\}$, by the form of the data, we clearly have

$$\|\phi\|_{L^\infty(\Sigma_{-r_0})} \lesssim \frac{\delta^{3/2}}{r_0}, \quad \|\psi\|_{L^\infty(\Sigma_{-r_0})} \lesssim \frac{\delta^{1/2}}{r_0}. \quad (3.1)$$

We will use Z or Z_j to denote any vector from $\{T, R_i, Q\}$ where $Q = t\underline{L}$. On Σ_{-r_0} , Z is simply ∂_r , Ω_i or $-r_0(\partial_t - \partial_r)$, therefore, we have

$$\|Z^m \psi\|_{L^\infty(\Sigma_{-r_0})} \lesssim \frac{\delta^{1/2-l}}{r_0}, \quad (3.2)$$

with l is the number of T 's and $Z \in \{T, \Omega_i, Q\}$. We remark that throughout the whole argument Q appears at most twice in the string of Z 's.

We also consider the incoming energy for $Z^m \psi$ on Σ_{-r_0} . According to (3.2), we have

$$\|\underline{L}(Z^m \psi)\|_{L^2(\Sigma_{-r_0})} + \|\not{d}(Z^m \psi)\|_{L^2(\Sigma_{-r_0})} \lesssim \frac{\delta^{1-l}}{r_0}, \quad \|T(Z^m \psi)\|_{L^2(\Sigma_{-r_0})} \lesssim \delta^{-l}$$

where \not{d} denotes for the exterior differential on $S_{t, \underline{u}}$. In terms of L , for $m \in \mathbb{Z}_{\geq 0}$, we obtain

$$\|\underline{L}(Z^m \psi)\|_{L^2(\Sigma_{-r_0})} + \|\not{d}(Z^m \psi)\|_{L^2(\Sigma_{-r_0})} \lesssim \frac{\delta^{1-l}}{r_0}, \quad \|L(Z^m \psi)\|_{L^2(\Sigma_{-r_0})} \lesssim \delta^{-l} \quad (3.3)$$

where l is the number of T 's in Z 's.

We also consider the estimates on some connection coefficients on Σ_{-r_0} . For μ , since we have $g(T, T) = c^{-2}\mu^2$ and $T = \partial_r$ on Σ_{-r_0} , we then have $\mu = c$ on Σ_{r_0} . Since $c = (1 + 2G''(0)(\partial_t \phi)^2)^{-\frac{1}{2}}$, according to (3.1), for sufficiently small δ , we obtain

$$\|\mu - 1\|_{L^\infty(\Sigma_{-r_0})} \lesssim \frac{\delta}{r_0^2}. \quad (3.4)$$

For $\underline{\chi}_{AB}$, since $\underline{\chi}_{AB} = -c\theta_{AB} = -\frac{c}{r_0}\not\!\theta_{AB}$, we have $\underline{\chi}_{AB} + \frac{1}{r_0}\not\!\theta_{AB} = (1-c)\frac{1}{r_0}\not\!\theta_{AB}$. Hence,

$$\|\underline{\chi}_{AB} + \frac{1}{r_0}\not\!\theta_{AB}\|_{L^\infty(\Sigma_{-r_0})} \lesssim \frac{\delta}{r_0^3}. \quad (3.5)$$

It measures the difference between the 2nd fundamental form with respect to $g_{\alpha\beta}$ and $m_{\alpha\beta}$.

3.2. Bootstrap assumptions and the main estimates. We expect the estimates (3.1), (3.2) and (3.3) hold not only for $t = -r_0$ but also for later time slice in W_δ^* . For this purpose, we will run a bootstrap argument to derive the *a priori* estimates for the $Z^m\psi$'s.

3.2.1. Conventions. We first introduce three large positive integers N_{top} , N_μ and N_∞ . They will be determined later on. We require that $N_\mu = \lfloor \frac{3}{4}N_{\text{top}} \rfloor$ and $N_\infty = \lfloor \frac{1}{2}N_{\text{top}} \rfloor + 1$. N_{top} will eventually be the total number of derivatives applied to the linearized equation $\square_{\bar{g}}\psi = 0$.

As in [18], to count the number of derivatives, we define the *order* of an object. The solution ϕ is considered as an order -1 object. The variations $\psi = A\phi$ are of order 0. The metric g depends only on ψ , so it is of order 0. The inverse density function μ is of order 0. The connection coefficients are 1st order derivatives on g , hence, of order 1. In particular, $\underline{\chi}_{AB}$ is of order 1. Let $\alpha = (i_1, \dots, i_{k-1})$ be a multi-index with i_j 's from $\{1, 2, 3\}$. We use $Z^\alpha\psi$ as a schematic expression of $Z_{i_1}Z_{i_2}\dots Z_{i_{k-1}}\psi$. The order of $Z^\alpha\psi$ is $|\alpha|$, where $|\alpha| = k-1$. Similarly, for any tensor of order $|\alpha|$, after taking m derivatives, its order becomes $|\alpha| + m$. The highest order objects in this paper will be of order $N_{\text{top}} + 1$.

Let $l \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}$. We use $\mathcal{O}_{k,j}^l$ or $\mathcal{O}_{k,j}^{\leq l}$ to denote any term of order l or at most l with estimates

$$\|\mathcal{O}_{k,j}^l\|_{L^\infty(\Sigma_t)} \lesssim \frac{\delta^{\frac{1}{2}k}}{(-t)^j}, \quad \|\mathcal{O}_{k,j}^{\leq l}\|_{L^\infty(\Sigma_t)} \lesssim \frac{\delta^{\frac{1}{2}k}}{(-t)^j}.$$

Similarly, we use $\Psi_{k,j}^l$ or $\Psi_{k,j}^{\leq l}$ to denote any term of order l or at most l with estimates

$$\|\Psi_{k,j}^l\|_{L^\infty(\Sigma_t)} \lesssim \frac{\delta^{\frac{1}{2}k}}{(-t)^j}, \quad \|\Psi_{k,j}^{\leq l}\|_{L^\infty(\Sigma_t)} \lesssim \frac{\delta^{\frac{1}{2}k}}{(-t)^j},$$

and moreover, it can be explicitly expressed a function of the variations ψ . For example, $\partial_t\phi \cdot \partial_i\phi \in \Psi_{1,1}^0$; A term of the form $\prod_{i=1}^n Z^{\alpha_i}\psi$ so that $\max|\alpha_i| \leq m$ is $\Psi_{n-2l,1}^{\leq m}$, where l is the number of T appearing in the derivatives. Note that $\underline{\chi}$ and μ can not be expressed explicitly in terms of ψ . The $\mathcal{O}_{k,j}^l$ terms (or similarly the $\Psi_{k,j}^l$ terms) obey the following algebraic rules:

$$\mathcal{O}_{k,j}^{\leq l} + \mathcal{O}_{k',j'}^{\leq l'} = \mathcal{O}_{\min(k,k'), \min(j,j')}^{\leq \max(l,l')}, \quad \mathcal{O}_{k,j}^{\leq l} \mathcal{O}_{k',j'}^{\leq l'} = \mathcal{O}_{k+k', j+j'}^{\leq \max(l,l')}.$$

3.2.2. Bootstrap assumptions on L^∞ norms. Motivated by (3.2), we make the following bootstrap assumptions (B.1) on W_δ^* : For all t and $2 \leq |\alpha| \leq N_\infty$,¹

$$\|\psi\|_{L^\infty(\Sigma_t)} + (-t)\|\underline{L}\psi\|_{L^\infty(\Sigma_t)} + (-t)\|\not\!\psi\|_{L^\infty(\Sigma_t)} + \delta\|T\psi\|_{L^\infty(\Sigma_t)} + \delta^l\|Z^\alpha\psi\|_{L^\infty(\Sigma_t)} \lesssim \delta^{1/2}M(-t)^{-1}. \quad (\text{B.1})$$

where l is the number of T 's appearing in Z^α and M is a large positive constant depending on ϕ . We will show that if δ is sufficiently small which may depend on M , then we can choose M in such a way that it depends only on the initial datum.

¹For a multi-index α , the symbol $\alpha - 1$ means another multi-index β with degree $|\beta| = |\alpha| - 1$.

3.2.3. *Energy norms.* For $(t, \underline{u}) \in [-r_0, t^*] \times [0, \delta]$, let $d\mu_{\mathcal{G}}$ be the volume form of \mathcal{G} . For a function $f(t, \underline{u}, \theta)$, we define

$$\int_{\Sigma_t^{\underline{u}}} f = \int_0^{\underline{u}} \left(\int_{S_{t, \underline{u}'}} f(t, \underline{u}', \theta) d\mu_{\mathcal{G}} \right) d\underline{u}', \quad \int_{\underline{C}_{\underline{u}}^t} f = \int_{-r_0}^t \left(\int_{S_{\tau, \underline{u}}} f(\tau, \underline{u}, \theta) d\mu_{\mathcal{G}} \right) d\tau.$$

For a function $\Psi(t, \underline{u}, \theta)$, we define the energy flux through the hypersurfaces $\Sigma_t^{\underline{u}}$ and $\underline{C}_{\underline{u}}^t$ as

$$\begin{aligned} E(\Psi)(t, \underline{u}) &= \int_{\Sigma_t^{\underline{u}}} (L\Psi)^2 + \mu \left((\underline{L}\Psi)^2 + (1 + \mu)|\not{d}\Psi|^2 \right), \quad F(\Psi)(t, \underline{u}) = \int_{\underline{C}_{\underline{u}}^t} (\underline{L}\Psi)^2 + \mu|\not{d}\Psi|^2, \\ \underline{E}(\Psi)(t, \underline{u}) &= (-t)^2 \int_{\Sigma_t^{\underline{u}}} \mu \left(\left(\underline{L}\Psi + \frac{1}{2} \widetilde{\text{tr}} \chi \Psi \right)^2 + |\not{d}\Psi|^2 \right), \quad \underline{F}(\Psi)(t, \underline{u}) = \int_{\underline{C}_{\underline{u}}^t} (-t')^2 \left(\underline{L}\Psi + \frac{1}{2} \widetilde{\text{tr}} \chi \Psi \right)^2. \end{aligned} \quad (3.6)$$

Here $\widetilde{\text{tr}} \chi$ is the trace of χ with respect to the induced conformal metric \widetilde{g} .

For each integer $0 \leq k \leq N_{\text{top}}$, we define

$$\begin{aligned} E_{k+1}(t, \underline{u}) &= \sum_{\psi} \sum_{|\alpha|=k-1} \delta^{2l} E(Z^{\alpha+1} \psi)(t, \underline{u}), \quad F_{k+1}(t, \underline{u}) = \sum_{\psi} \sum_{|\alpha|=k-1} \delta^{2l} F(Z^{\alpha+1} \psi)(t, \underline{u}), \\ \underline{E}_{k+1}(t, \underline{u}) &= \sum_{\psi} \sum_{|\alpha|=k-1} \delta^{2l} \underline{E}(Z^{\alpha+1} \psi)(t, \underline{u}), \quad \underline{F}_{k+1}(t, \underline{u}) = \sum_{\psi} \sum_{|\alpha|=k-1} \delta^{2l} \underline{F}(Z^{\alpha+1} \psi)(t, \underline{u}), \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} E_{\leq k+1}(t, \underline{u}) &:= \sum_{j \leq k} E_{j+1}(t, \underline{u}), \quad \underline{E}_{\leq k+1}(t, \underline{u}) := \sum_{j \leq k} \underline{E}_{j+1}(t, \underline{u}) \\ F_{\leq k+1}(t, \underline{u}) &:= \sum_{j \leq k} F_{j+1}(t, \underline{u}), \quad \underline{F}_{\leq k+1}(t, \underline{u}) := \sum_{j \leq k} \underline{F}_{j+1}(t, \underline{u}) \end{aligned} \quad (3.8)$$

where l is the number of T 's appearing in Z^{α} . The symbol \sum_{ψ} means to sum over all the first order variations $A\phi$ of ψ . For the sake of simplicity, we shall omit this sum symbol in the sequel.

For each integer $0 \leq k \leq N_{\text{top}}$, we assign a nonnegative integer b_k to k in such a way that

$$b_0 = b_1 = \dots = b_{N_{\mu}} = 0, \quad b_{N_{\mu}+1} < b_{N_{\mu}+2} < \dots < b_{N_{\text{top}}}. \quad (3.9)$$

We call b_k 's *the blow-up indices*. The sequence $(b_k)_{0 \leq k \leq N_{\text{top}}}$ will be determined later on.

For each integer $0 \leq k \leq N_{\text{top}}$, we also define the modified energy $\widetilde{E}_k(t, \underline{u})$ and $\widetilde{\underline{E}}_k(t, \underline{u})$ as

$$\begin{aligned} \widetilde{E}_{k+1}(t, \underline{u}) &= \sup_{\tau \in [-r_0, t]} \{ \mu_m^{\underline{u}}(\tau)^{2b_{k+1}} E_{k+1}(\tau, \underline{u}) \}, \quad \widetilde{\underline{E}}_{k+1}(t, \underline{u}) = \sup_{\tau \in [-r_0, t]} \{ \mu_m^{\underline{u}}(\tau)^{2b_{k+1}} \underline{E}_{k+1}(\tau, \underline{u}) \} \\ \widetilde{F}_{k+1}(t, \underline{u}) &= \sup_{\tau \in [-r_0, t]} \{ \mu_m^{\underline{u}}(\tau)^{2b_{k+1}} F_{k+1}(\tau, \underline{u}) \}, \quad \widetilde{\underline{F}}_{k+1}(t, \underline{u}) = \sup_{\tau \in [-r_0, t]} \{ \mu_m^{\underline{u}}(\tau)^{2b_{k+1}} \underline{F}_{k+1}(\tau, \underline{u}) \} \\ \widetilde{E}_{\leq k+1}(t, \underline{u}) &:= \sum_{j \leq k} \widetilde{E}_{j+1}(t, \underline{u}), \quad \widetilde{\underline{E}}_{\leq k+1}(t, \underline{u}) := \sum_{j \leq k} \widetilde{\underline{E}}_{j+1}(t, \underline{u}), \\ \widetilde{F}_{\leq k+1}(t, \underline{u}) &:= \sum_{j \leq k} \widetilde{F}_{j+1}(t, \underline{u}), \quad \widetilde{\underline{F}}_{\leq k+1}(t, \underline{u}) := \sum_{j \leq k} \widetilde{\underline{F}}_{j+1}(t, \underline{u}), \end{aligned} \quad (3.10)$$

where $\mu_m^{\underline{u}}(t)$ is defined as

$$\mu_m^{\underline{u}}(t) = \min_{\tau \in [-r_0, t]} \left\{ \inf_{(\underline{u}', \theta) \in [0, \underline{u}] \times \mathbb{S}^2} \mu(\tau, \underline{u}', \theta), 1 \right\} \quad (3.11)$$

For simplicity, we omit the parameter \underline{u} to write the weight as $\mu_m(t)$. When we work on the estimates on $W_{\underline{u}}^t$, the weight $\mu_m(t)$ means $\mu_m^{\underline{u}}(t)$.

We now state the main estimates of the paper.

Theorem 3.1. *There exists a constant δ_0 depending only on the seed data ϕ_0 and ϕ_1 , so that for all $\delta < \delta_0$, there exist constants M_0 , N_{top} and $(b_k)_{0 \leq k \leq N_{top}}$ with the following properties*

- M_0 , N_{top} and $(b_k)_{0 \leq k \leq N_{top}}$ depend only on the initial datum.
- The inequalities (B.1) holds for all $t < t^*$ with $M = M_0$.
- Either $t^* = -1$ and we have a smooth solution in the time slab $[-r_0, -1]$; or $t^* < -1$ and then ψ_α 's as well as the rectangular coordinates x^i 's extend smoothly as functions of the coordinates $(t, \underline{u}, \theta)$ to $t = t^*$ and there is at least one point on $\Sigma_{t^*}^\delta$ where μ vanishes, thus we have shock formation.
- If, moreover, the initial data satisfies the largeness condition (1.14), then in fact $t^* < -1$.

3.3. Preliminary estimates based on (B.1). In this subsection, we list the pointwise estimates for components of optical metric, connection coefficients as well as deformation tensors associated to various vectorfields based on (B.1). The proofs are similar to those in [18].

3.3.1. Estimates on metric and connection.

Lemma 3.2. *For sufficiently small² δ , we have*

$$\begin{aligned} \|c - 1\|_{L^\infty(\Sigma_t)} &\lesssim \delta M^2(-t)^{-2}, \quad \frac{1}{2} \leq c \leq 2, \\ (-t)\|\underline{L}c\|_{L^\infty(\Sigma_t)} + (-t)\|\not{D}c\|_{L^\infty(\Sigma_t)} &\lesssim \delta M^2(-t)^{-2}, \quad \|Tc\|_{L^\infty(\Sigma_t)} \lesssim M^2(-t)^{-2}. \end{aligned} \quad (3.12)$$

Lemma 3.3. *For sufficiently small δ , we have*

$$\|m\|_{L^\infty(\Sigma_t)} + (-t)\|\not{D}m\|_{L^\infty(\Sigma_t)} \lesssim M^2(-t)^{-2}, \quad \|Tm\|_{L^\infty(\Sigma_t)} \lesssim \delta^{-1}M^2(-t)^{-2}, \quad (3.13)$$

$$\|e\|_{L^\infty(\Sigma_t)} + (-t)\|\not{D}e\|_{L^\infty(\Sigma_t)} \lesssim \delta M^2(-t)^{-2}, \quad \|Te\|_{L^\infty(\Sigma_t)} \lesssim M^2(-t)^{-2}, \quad (3.14)$$

$$\|\mu - 1\|_{L^\infty(\Sigma_t)} + (-t)\|\underline{L}\mu\|_{L^\infty(\Sigma_t)} \lesssim M^2(-t)^{-1}. \quad (3.15)$$

Lemma 3.4. *For sufficiently small δ , we have*

$$\|T\mu\|_{L^\infty(\Sigma_t)} \lesssim \delta^{-1}M^2(-t)^{-1}, \quad (-t)\|\not{D}\mu\|_{L^\infty(\Sigma_t)} \lesssim M^2(-t)^{-1}. \quad (3.16)$$

As a corollary of these lemmas, we can show

Corollary 3.5. *For sufficiently small δ , we have*

$$\|\underline{\zeta}\|_{L^\infty(\Sigma_t)} \lesssim \delta M^2(-t)^{-2}, \quad \|\underline{\eta}\|_{L^\infty(\Sigma_t)} \lesssim M^2(-t)^{-2}. \quad (3.17)$$

We now estimate $\underline{\chi}_{AB}$. For this purpose, we introduce

$$\underline{\chi}'_{AB} = \underline{\chi}_{AB} + \frac{\not{g}_{AB}}{\underline{u} - t}. \quad (3.18)$$

which measures the deviation of $\underline{\chi}_{AB}$ from the null 2nd fundamental form in Minkowski space. We have

Lemma 3.6. *For sufficiently small δ , we have*

$$\|\underline{\chi}'_{AB}\|_{L^\infty(\Sigma_t)} \lesssim \delta M^2(-t)^{-3}. \quad (3.19)$$

²This sentence always means that, there exists $\varepsilon = \varepsilon(M)$ so that for all $\delta \leq \varepsilon$, we have ...

Proof. According to (2.35), we have

$$\underline{L}(\underline{\chi}'_{AB}) = e\underline{\chi}'_{AB} + \underline{\chi}'_A{}^C \underline{\chi}'_{BC} - \frac{e}{\underline{u}-t} \not{g}_{AB} - \underline{\alpha}'_{AB}. \quad (3.20)$$

Hence, $\underline{L}|\underline{\chi}'|^2_g = 2e|\underline{\chi}'|^2 - 2\underline{\chi}'_A{}^B \underline{\chi}'_B{}^C \underline{\chi}'_C{}^A + \frac{2|\underline{\chi}'|^2}{\underline{u}-t} - \frac{2e}{\underline{u}-t} \text{tr}\underline{\chi}' - 2\underline{\chi}'^{AB} \underline{\alpha}_{AB}$. Therefore, we obtain

$$\underline{L}((t-\underline{u})^2|\underline{\chi}'|) \lesssim (t-\underline{u})^2 \left((|e| + |\underline{\chi}'|)|\underline{\chi}'| + \frac{|e|}{\underline{u}-t} + |\underline{\alpha}'| \right) \quad (3.21)$$

Let $\mathcal{P}(t)$ be the property that $\|\underline{\chi}'\|_{L^\infty(\Sigma_{t'})} \leq C_0 \delta M^2(-t')^{-3}$ for all $t' \in [-r_0, t]$. By choosing C_0 suitably large, according to the assumptions on initial data, we have $\|\underline{\chi}'\|_{L^\infty(\Sigma_{-r_0})} < C_0 \delta r_0^{-3}$. It follows by continuity that $\mathcal{P}(t)$ is true for t sufficiently close to $-r_0$. Let t_0 be the upper bound of $t \in [-r_0, t_0]$ for which $\mathcal{P}(t)$ holds. By continuity, $\mathcal{P}(t_0)$ is true. Therefore, for $t \leq t_0$, we have $|\underline{\chi}'(t)| + |e(t)| \leq (C_0 + C_1) \delta M^2(-t)^{-3}$ for a universal constant C_1 . According to the explicit formula of $\underline{\alpha}'$ and (B.1), for sufficiently small δ , there is a universal constant C_3 so that $\|\underline{\alpha}'\|_{L^\infty(\Sigma_t)} \leq C_3 \delta M^2(-t)^{-4}$. In view of (3.21), there is a universal constant C_4 so that

$$\underline{L}((t-\underline{u})^2|\underline{\chi}'|) \leq C_4(t-\underline{u})^2 ((C_0 + C_1) \delta M^2(-t)^{-3} |\underline{\chi}'| + (C_1 + C_3) \delta M^2(-t)^{-4}). \quad (3.22)$$

If we define $x(t) = (t-\underline{u})^2|\underline{\chi}'|$ along the integral curve of \underline{L} , then we can rewrite (3.22) as $\frac{dx}{dt} \leq fx + g$, where $f(t) = C_4(C_0 + C_1) \delta M^2(-t)^{-3}$ and $g(t) = C_4(C_1 + C_3) \delta M^2(-t)^{-2}$. By integrating from $-r_0$ to t , we obtain

$$x(t) \leq e^{\int_{-r_0}^t f(t') dt'} (x(-r_0) + \int_{-r_0}^t e^{-\int_{-r_0}^{t'} f(t'') dt''} g(t') dt').$$

Taking into account the facts that $\int_{-r_0}^t f(t') dt' \leq C_4(C_0 + C_1) \delta M^2(-t)^{-1}$ and $\int_{-r_0}^t g(t') dt' \leq C_4(C_1 + C_3) \delta M^2(-t)^{-1}$, since $(t-\underline{u})^2 \sim t^2$ on the support of $\underline{\chi}'$, for some universal constant C_5 , we have

$$\begin{aligned} \|\underline{\chi}'\|_{L^\infty(\Sigma_t)} &\leq C_5 e^{C_4(C_0+C_1)\delta M^2(-t)^{-2}} \left(r_0^2 \|\underline{\chi}'\|_{L^\infty(\Sigma_{-r_0})} + C_4(C_1 + C_3) \delta M^2(-t)^{-1} \right) \\ &\leq C_5 e^{C_4(C_0+C_1)\delta M^2(-t)^{-3}} \left(r_0^3 \|\underline{\chi}'\|_{L^\infty(\Sigma_{-r_0})} + C_4(C_1 + C_3) \delta M^2 \right) \end{aligned} \quad (3.23)$$

We then fix C_0 in such a way that $C_0 > 2C_5C_4(C_1 + C_3)$ and $C_0 > \frac{r_0^3 \|\underline{\chi}'\|_{L^\infty(\Sigma_{-r_0})}}{4C_5}$. Provided δ satisfying $\delta < \frac{\log 2}{C_4(C_0+C_1)M^2}$, the estimate (3.23) implies $\|\underline{\chi}'\|_{L^\infty(\Sigma_t)} < C_0 \delta M^2(-t)^{-3}$ for all $t \in [-r_0, t_0]$. By continuity, $\mathcal{P}(t)$ holds for some $t > t_0$. Hence the lemma follows. \square

Remark 3.7 (Estimates related to the conformal optical metric \tilde{g}). As in [18], we shall use \sim to indicate the quantities defined with respect to \tilde{g} .

We expect the quantities (with \sim) defined with respect to \tilde{g} have the similar estimates as the counterparts (without \sim) defined with respect to g . This is clear: the difference can be explicitly computed in terms of c and hence controlled by the estimates on c . For example, the difference between $\tilde{\chi}'_{AB}$ and $\underline{\chi}'_{AB}$ is $\tilde{\chi}_{AB} - \underline{\chi}_{AB} = -\frac{1}{2c^2} \underline{L}(c) \not{g}_{AB}$, based on (3.12) and (3.19), we have

$$\left\| \tilde{\chi}_{AB} + \frac{\not{g}_{AB}}{\underline{u}-t} \right\|_{L^\infty(\Sigma_t)} \lesssim \delta M^2(-t)^{-3}. \quad (3.24)$$

3.3.2. Estimates on deformation tensors. Now we consider the deformation tensors of the following five commutation vectorfields: $Z_1 = T, Z_2 = R_1, Z_3 = R_2, Z_4 = R_3, Z_5 = Q$. The notation Z^α for a multi-index $\alpha = (i_1, \dots, i_m)$ means $Z_{i_1} Z_{i_2} \dots Z_{i_m}$ with $i_j \in \{1, 2, 3, 4, 5\}$. For a vectorfield Z , the deformation tensor $^{(Z)}\pi$ or $^{(Z)}\tilde{\pi}$ with respect to g and \tilde{g} is defined by $^{(Z)}\pi_{\alpha\beta} = \nabla_\alpha Z_\beta + \nabla_\beta Z_\alpha$ or $^{(Z)}\tilde{\pi}_{\alpha\beta} = \frac{1}{c}^{(Z)}\pi_{\alpha\beta} + Z(\frac{1}{c})g_{\alpha\beta}$. As in

[18], a direct calculation gives the expressions for deformation tensors of T

$$\begin{aligned} {}^{(T)}\tilde{\pi}_{\underline{LL}} &= 0, \quad {}^{(T)}\tilde{\pi}_{LL} = 4c^{-1}\mu T(c^{-2}\mu), \quad {}^{(T)}\tilde{\pi}_{\underline{LL}} = -2c^{-1}(T\mu - c^{-1}\mu T(c)), \\ {}^{(T)}\tilde{\pi}_{LA} &= -c^{-3}\mu(\underline{\zeta}_A + \underline{\eta}_A), \quad {}^{(T)}\tilde{\pi}_{\underline{LA}} = -c^{-1}(\underline{\zeta}_A + \underline{\eta}_A), \\ {}^{(T)}\hat{\tilde{\pi}}_{AB} &= -2c^{-3}\mu\hat{\chi}_{AB}, \quad \text{tr}^{(T)}\tilde{\pi} = -2c^{-3}\mu\text{tr}_{\tilde{g}}\tilde{\chi}. \end{aligned} \quad (3.25)$$

and those of Q :

$$\begin{aligned} {}^{(Q)}\tilde{\pi}_{\underline{LL}} &= 0, \quad {}^{(Q)}\tilde{\pi}_{LL} = 4t\underline{L}(c^{-2}\mu)(c^{-1}\mu) - 4c^{-3}\mu^2, \quad {}^{(Q)}\tilde{\pi}_{\underline{LL}} = -2t\underline{L}(c^{-1}\mu) - 2c^{-1}\mu \\ {}^{(Q)}\tilde{\pi}_{LA} &= 0, \quad {}^{(Q)}\tilde{\pi}_{LA} = 2tc^{-1}(\underline{\zeta}_A + \underline{\eta}_A), \quad {}^{(Q)}\hat{\tilde{\pi}}_{AB} = 2tc^{-1}\hat{\chi}_{AB}, \quad \text{tr}^{(Q)}\tilde{\pi} = 2c^{-1}t\text{tr}_{\tilde{g}}\tilde{\chi}. \end{aligned} \quad (3.26)$$

as well as their estimates:

$$\begin{aligned} \|\mu^{-1}{}^{(T)}\tilde{\pi}_{LL}\|_{L^\infty(\Sigma_t)} &\lesssim \delta^{-1}M^2(-t)^{-1}, \quad \|{}^{(T)}\tilde{\pi}_{\underline{LL}}\|_{L^\infty(\Sigma_t)} \lesssim \delta^{-1}M^2(-t)^{-1}, \\ \|\mu^{-1}{}^{(T)}\tilde{\pi}_{LA}\|_{L^\infty(\Sigma_t)} &\lesssim M^2(-t)^{-2}, \quad \|{}^{(T)}\tilde{\pi}_{\underline{LA}}\|_{L^\infty(\Sigma_t)} \lesssim M^2(-t)^{-2}, \\ \|\mu^{-1}{}^{(T)}\hat{\tilde{\pi}}_{AB}\|_{L^\infty(\Sigma_t)} &\lesssim \delta M^2(-t)^{-3}, \quad \|\mu^{-1}\text{tr}^{(T)}\tilde{\pi}\|_{L^\infty(\Sigma_t)} \lesssim (-t)^{-1}. \end{aligned} \quad (3.27)$$

$$\begin{aligned} \|\mu^{-1}{}^{(Q)}\tilde{\pi}_{LL}\|_{L^\infty(\Sigma_t)} &\lesssim 1 + M^2(-t)^{-1}, \quad \|{}^{(Q)}\tilde{\pi}_{\underline{LL}}\|_{L^\infty(\Sigma_t)} \lesssim 1 + M^2(-t)^{-1}, \quad \|{}^{(Q)}\tilde{\pi}_{LA}\|_{L^\infty(\Sigma_t)} \lesssim M^2(-t)^{-1}, \\ \|\mu^{-1}{}^{(Q)}\hat{\tilde{\pi}}_{AB}\|_{L^\infty(\Sigma_t)} &\lesssim \delta M^2(-t)^{-2}, \quad \|\text{tr}^{(Q)}\tilde{\pi}\|_{L^\infty(\Sigma_t)} \lesssim 1. \end{aligned} \quad (3.28)$$

Actually the estimate for $\text{tr}^{(Q)}\tilde{\pi}$ can be improved more precisely. Let us rewrite the following component of deformation tensor of Q :

$$\text{tr}^{(Q)}\tilde{\pi} = 2c^{-1}t\text{tr}_{\tilde{g}}\tilde{\chi} = 2c^{-1}t\text{tr}_{\tilde{g}}\tilde{\chi}' + 4\frac{c^{-1}\underline{u}}{\underline{u}-t} - 4(c^{-1}-1) - 4$$

This tells us:

$$\|\text{tr}^{(Q)}\tilde{\pi} + 4\|_{L^\infty(\Sigma_t^{\frac{u}{t}})} \lesssim \delta(-t)^{-1}. \quad (3.29)$$

To derive the pointwise estimates for the deformation tensors of R_i requires more work. First, in view of (2.40), we have the following expressions:

$$\begin{aligned} {}^{(R_i)}\pi_{\underline{LL}} &= 0, \quad {}^{(R_i)}\pi_{TT} = 2c^{-1}\mu \cdot R_i(c^{-1}\mu), \quad {}^{(R_i)}\pi_{\underline{LT}} = -R_i(\mu), \quad {}^{(R_i)}\pi_{AB} = -2\lambda_i\theta_{AB}, \\ {}^{(R_i)}\pi_{TA} &= -c^{-1}\mu(\theta_{AB} - \frac{\not{g}_{AB}}{\underline{u}-t})R_i^B + c^{-1}\mu\varepsilon_{ikj}y^kX_A^j + \lambda_iX_A(c^{-1}\mu), \\ {}^{(R_i)}\pi_{LA} &= -\chi_{AB}R_i^B + \underline{L}^k\varepsilon_{ikj}X_A^j + c\mu^{-1}\lambda_i\underline{\zeta}_A = -(\chi_{AB} + \frac{\not{g}_{AB}}{\underline{u}-t})R_i^B + \varepsilon_{ikj}z^kX_A^j + c\mu^{-1}\lambda_i\underline{\zeta}_A. \end{aligned} \quad (3.30)$$

The Latin indices i, j, k are defined with respect to the Cartesian coordinates on Σ_t . To bound deformation of R_i , it suffices to control the λ_i 's, y^i 's and z^i 's.

First of all, we have

$$|\hat{T}^i| + |\underline{L}^i| \lesssim 1. \quad (3.31)$$

The proof is straightforward: $g|_{\Sigma_t}$ is flat and \hat{T} is the unit normal of $S_{t,\underline{u}}$ in Σ_t , so $|\hat{T}^i| \leq 1$. In the Cartesian coordinates (t, x^1, x^2, x^3) , $\underline{L} = \partial_t - c\hat{T}^i\partial_i$, so $|\underline{L}^i| \lesssim 1$.

Let $r = (\sum_{i=1}^3 x^i)^{\frac{1}{2}}$. Since $Tr = c^{-1}\mu \sum_{i=1}^3 \frac{x^i \hat{T}^i}{r}$, (3.31) implies that $|Tr| \lesssim 1 + M^2(-t)^{-1}$. We then integrate from 0 to \underline{u} , since $r = -t$ when $\underline{u} = 0$ and $|\underline{u}| \leq \delta$, we obtain $|r+t| \lesssim \delta M^2$. In application, for sufficiently small δ , we often use $r \sim |t|$. The estimate can also be written as

$$\left| \frac{1}{r} - \frac{1}{\underline{u} + |t|} \right| \lesssim \delta M^2(-t)^{-2}. \quad (3.32)$$

To control λ_i , we consider its \underline{L} derivative. By definition $\lambda_i = g(\Omega_i, \hat{T})$, we can write its derivative along \underline{L} as $\underline{L}\lambda_i = \sum_{k=1}^3 (\Omega_i)^k \underline{L}\hat{T}^k = \sum_{k=1}^3 (\Omega_i)^k X^A(c) X_A^k$. As $|t| \sim r$, we have $|\Omega_i| \lesssim |t|$, this implies

$$\|\underline{L}\lambda_i\|_{L^\infty(\Sigma_t)} \lesssim \delta M^2(-t)^{-2}. \quad (3.33)$$

Since $\lambda_i = 0$ on Σ_{-r_0} , we have

$$\|\lambda_i\|_{L^\infty(\Sigma_t)} \lesssim \delta M^2(-t)^{-1}. \quad (3.34)$$

To control y^i 's and z^i 's, let $\bar{y} = (y^1, y^2, y^3)$ and $\bar{x} = (x^1, x^2, x^3)$, we then have

$$|\bar{y} - (\frac{1}{r} - \frac{1}{\underline{u} - t})\bar{x}|^2 = |(g(\hat{T}, \partial_r) - 1)\partial_r|^2 + \frac{1}{r^2} \sum_{i=1}^3 \lambda_i^2 \quad (3.35)$$

On the other hand, we have $1 - |g(\hat{T}, \partial_r)|^2 = \frac{1}{r^2} \sum_{i=1}^3 \lambda_i^2 \lesssim \delta M^2(-t)^{-4}$. While on $S_{t,0}$, Since $g(\partial_r, \hat{T}) = 1$ on $S_{t,0}$, for sufficiently small δ , the angle between ∂_r and \hat{T} is less than $\frac{\pi}{2}$, which implies $1 + g(\hat{T}, \partial_r) \geq 1$. Therefore,

$$|1 - g(\hat{T}, \partial_r)| \lesssim \delta M^2(-t)^{-4}. \quad (3.36)$$

Together with (3.34) and (3.35), this implies

$$|y^i| \lesssim \delta M^2(-t)^{-2}, \quad |y^i| \lesssim \delta M^2(-t)^{-2}. \quad (3.37)$$

We then control z^i from its definition

$$|z^i| \lesssim \delta M^2(-t)^{-2}. \quad (3.38)$$

The derivatives of λ_i on Σ_t are given by $X_A(\lambda_i) = \left(\theta_{AB} - \frac{\theta_{AB}}{\underline{u}-t}\right) R_i^B - \varepsilon_{ikj} y^k X_A^j$ and $T(\lambda_i) = -R_i(c^{-1}\mu)$. Hence,

$$\|\not\partial\lambda_i\|_{L^\infty(\Sigma_t)} \lesssim \delta M^2(-t)^{-2}, \quad \|T\lambda_i\|_{L^\infty(\Sigma_t)} \lesssim M^2(-t)^{-1}$$

Finally, we obtain the following estimates for the deformation tensor of R_i :

$$\begin{aligned} {}^{(R_i)}\pi_{\underline{L}\underline{L}} &= 0, \quad \|{}^{(R_i)}\tilde{\pi}_{\underline{L}T}\|_{L^\infty(\Sigma_t)} \lesssim M^2(-t)^{-1}, \quad \|\mu^{-1(R_i)}\pi_{TT}\|_{L^\infty(\Sigma_t)} \lesssim M^2(-t)^{-1}, \\ \|{}^{(R_i)}\pi_{\underline{L}A}\|_{L^\infty(\Sigma_t)} &\lesssim \delta M^2(-t)^{-2}, \quad \|{}^{(R_i)}\pi_{TA}\|_{L^\infty(\Sigma_t)} \lesssim \delta M^4(-t)^{-2}, \\ \|{}^{(R_i)}\hat{\pi}_{AB}\|_{L^\infty(\Sigma_t)} &\lesssim \delta^2 M^4(-t)^{-4}, \quad \|\text{tr}^{(R_i)}\not\pi\|_{L^\infty(\Sigma_t)} \lesssim \delta M^4(-t)^{-2}. \end{aligned}$$

We use the relation $L = c^{-2}\mu\underline{L} + 2T$ to rewrite the above estimates in null frame as follows:

$$\begin{aligned} \|{}^{(R_i)}\pi_{\underline{L}\underline{L}}\|_{L^\infty(\Sigma_t)} &\lesssim M^2(-t)^{-1}, \quad \|\mu^{-1(R_i)}\pi_{LL}\|_{L^\infty(\Sigma_t)} \lesssim M^2(-t)^{-1}, \\ \|{}^{(R_i)}\pi_{\underline{L}A}\|_{L^\infty(\Sigma_t)} &\lesssim \delta M^2(-t)^{-2}, \quad \|{}^{(R_i)}\pi_{LA}\|_{L^\infty(\Sigma_t)} \lesssim \delta M^4(-t)^{-2}, \\ \|{}^{(R_i)}\hat{\pi}_{AB}\|_{L^\infty(\Sigma_t)} &\lesssim \delta^2 M^4(-t)^{-4}, \quad \|\text{tr}^{(R_i)}\not\pi\|_{L^\infty(\Sigma_t)} \lesssim \delta M^4(-t)^{-2}. \end{aligned} \quad (3.39)$$

The deformation tensors of R_i 's with respect to \tilde{g} are estimated by

$$\begin{aligned} {}^{(R_i)}\tilde{\pi}_{\underline{L}\underline{L}} &= 0, \quad \|{}^{(R_i)}\tilde{\pi}_{\underline{L}L}\|_{L^\infty(\Sigma_t)} \lesssim M^2(-t)^{-1}, \quad \|\mu^{-1(R_i)}\tilde{\pi}_{LL}\|_{L^\infty(\Sigma_t)} \lesssim M^2(-t)^{-1}, \\ \|{}^{(R_i)}\tilde{\pi}_{\underline{L}A}\|_{L^\infty(\Sigma_t)} &\lesssim \delta M^2(-t)^{-2}, \quad \|{}^{(R_i)}\tilde{\pi}_{LA}\|_{L^\infty(\Sigma_t)} \lesssim \delta M^4(-t)^{-2}, \\ \|{}^{(R_i)}\hat{\pi}_{AB}\|_{L^\infty(\Sigma_t)} &\lesssim \delta^2 M^4(-t)^{-4}, \quad \|\text{tr}^{(R_i)}\tilde{\not\pi}\|_{L^\infty(\Sigma_t)} \lesssim \delta M^4(-t)^{-2}. \end{aligned} \quad (3.40)$$

3.3.3. *Applications of the estimates on λ_i and y^i .* As in [18], based on calculations in [Ch-Shocks] and [Ch-Miao], we are able to show that the R_i derivatives are equivalent to the $(-t)\not{d}$ and $(-t)\nabla$ derivative. For a 1-form ξ on $S_{t,\underline{u}}$, we have $\sum_{i=1}^3 \xi(R_i)^2 = r^2(|\xi|^2 - (\xi(y'))^2)$. This is indeed can be derived from the formula $\sum_{i=1}^3 (R_i)^a (R_i)^b = r^2(\delta_{cd} - y'^c y'^d) \Pi_c^a \Pi_d^b$, where $a, b, c, d \in \{1, 2, 3\}$. In view of (3.32), (3.37) and the definition of y'^i , for sufficiently small δ , we have $\sum_{i=1}^3 \xi(R_i)^2 \sim r^2|\xi|^2$. Since r is bounded below and above by $(-t)$, we obtain $\sum_{i=1}^3 \xi(R_i)^2 \sim t^2|\xi|^2$. Similarly, for a k -covariant tensor ξ on $S_{t,\underline{u}}$, we have $\sum_{i_1, i_2, \dots, i_k=1}^3 \xi(R_{i_1}, R_{i_2}, \dots, R_{i_k})^2 \sim t^2|\xi|^2$. In particular, we can take $\xi = \not{d}\psi$, therefore, $\sum_{i=1}^3 (R_i\psi)^2 \sim t^2|\not{d}\psi|^2$. Henceforth, we omit the summation and write schematically as $|R_i\psi| \sim (-t)|\not{d}\psi|$.

We can also compare the R_i -derivatives with the ∇ -derivatives for tensors. For $S_{t,\underline{u}}$ -tangential 1-form ξ and vectorfield X , let $\not{L}_{R_i}\xi$ be the orthogonal projection of the Lie derivative $\mathcal{L}_{R_i}\xi$ onto the surface $S_{t,\underline{u}}$. Since $(\not{L}_{R_i}\xi)(X) = (\nabla_{R_i}\xi)(X) + \xi(\nabla_X R_i)$, we obtain

$$\sum_{i=1}^3 |\not{L}_{R_i}\xi|^2 = \sum_{i=1}^3 |\xi(R_i)|^2 + 2 \sum_{i=1}^3 \xi^k (\nabla_{R_i}\xi)_a (\nabla R_i)^a{}_k + \sum_{i=1}^3 \xi^k \xi^l (\nabla R_i)_{ak} (\nabla R_i)^a{}_l.$$

We also have $\sum_{i=1}^3 |\nabla_{R_i}\xi|^2 = r^2(\delta^{cd} - y'^c y'^d) (\nabla\xi)_{ca} (\nabla\xi)^a{}_d$. In view of the estimates on y'^i , for sufficiently small δ , we obtain

$$\sum_{i=1}^3 |\nabla_{R_i}\xi|^2 \gtrsim r^2 |\nabla\xi|^2.$$

Let ε_{ijk} be the volume form on Σ_t and v_i be a $S_{t,\underline{u}}$ 1-form with rectangular components $(v_i)_a = \Pi_a^b \varepsilon_{ibk} \xi_k$. By virtue of the formula $(\nabla R_i)_l^k = \Pi_k^m \Pi_l^n \varepsilon_{inm} - \lambda_i \theta_{kl}$, we have

$$\sum_{i=1}^3 \xi^k \xi^l (\nabla R_i)_{ak} (\nabla R_i)^a{}_l = |\xi|^2 + 2c \sum_{i=1}^3 \lambda_i \xi \cdot \underline{\chi} \cdot v_i + c^2 |\underline{\chi} \cdot \xi|^2 \sum_{i=1}^3 \lambda_i^2.$$

In view of the estimates on λ_i , for sufficiently small δ , we have $\sum_{i=1}^3 \xi^k \xi^l (\nabla R_i)_{ak} (\nabla R_i)^a{}_l \sim |\xi|^2$. Similarly, we have $|\sum_{i=1}^3 \xi^k (\nabla R_i)_a (\nabla R_i)^b{}_k| \lesssim r|\xi||\nabla\xi|$. Finally, we conclude that

$$|\xi|^2 + r^2 |\nabla\xi|^2 \lesssim \sum_{i=1}^3 |\not{L}_{R_i}\xi|^2 \lesssim |\xi|^2 + r^2 |\nabla\xi|^2.$$

Henceforth, we omit the summation and write schematically as $|\not{L}_{R_i}\xi| \sim |\xi| + (-t)|\nabla\xi|$. Similarly, for a tracefree symmetric 2-tensors θ_{AB} tangential to $S_{t,\underline{u}}$, we have $|\theta| + |\nabla\theta| \lesssim |\not{L}_{R_i}\theta| \lesssim |\theta| + (-t)|\nabla\theta|$. This will be applied to $\theta = \hat{\chi}_{AB}$ later on.

3.3.4. *Sobolev inequalities and elliptic estimates.* To obtain the Sobolev inequalities on $S_{t,\underline{u}}$, we introduce

$$I(t, \underline{u}) = \sup_{\substack{U \in S_{t,\underline{u}}, \\ \partial U \text{ is } C^1}} \frac{\min(|U|, |S_{t,\underline{u}} - U|)}{|\partial U|^2}$$

the isoperimetric constant on $S_{t,\underline{u}}$, where $|U|$, $|S_{t,\underline{u}} - U|$ and $|\partial U|$ are the measures of the corresponding sets with respect to \not{g} on $S_{t,\underline{u}}$. Therefore, in view of the fact that $R_i \sim r\nabla$, for sufficiently small δ , we have the following Sobolev inequalities:

$$\begin{aligned} \|f\|_{W^{1,4}(S_{t,\underline{u}})} &\lesssim |I(t, \underline{u})|^{\frac{1}{4}} |S_{t,\underline{u}}|^{-\frac{3}{4}} (\|f\|_{L^2(S_{t,\underline{u}})} + \|R_i f\|_{L^2(S_{t,\underline{u}})} + \|R_i R_j f\|_{L^2(S_{t,\underline{u}})}), \\ \|f\|_{L^\infty(S_{t,\underline{u}})} &\lesssim |I(t, \underline{u})|^{\frac{3}{4}} |S_{t,\underline{u}}|^{-\frac{1}{2}} (\|f\|_{L^2(S_{t,\underline{u}})} + \|R_i f\|_{L^2(S_{t,\underline{u}})} + \|R_i R_j f\|_{L^2(S_{t,\underline{u}})}). \end{aligned} \quad (3.41)$$

where $\|f\|_{W^{1,4}(S_{t,\underline{u}})}$ is defined as $\|f\|_{W^{1,4}(S_{t,\underline{u}})} = |S_{t,\underline{u}}|^{-1/2} \|f\|_{L^4(S_{t,\underline{u}})} + \|\not{d}f\|_{L^4(S_{t,\underline{u}})}$. It remains to control the isoperimetric constant $I(t, \underline{u})$.

We use T to generate a diffeomorphism of $S_{t,\underline{u}}$ to $S_{t,0}$ which maps U , $S_{t,\underline{u}} - U$ and ∂U to corresponding sets $U_{\underline{u}}$, $S_{t,0} - U_{\underline{u}}$ and $\partial U_{\underline{u}}$ on $S_{t,0}$. Let $U_{\underline{u}'}$, $S_{t,\underline{u}'} - U_{\underline{u}'}$, $\partial U_{\underline{u}'}$ be the inverse images of these on each $S_{t,\underline{u}'}$ for $\underline{u}' \in [0, \underline{u}]$. Since $\mathcal{L}_T \mathcal{G}_{AB} = 2c^{-1}\mu\theta = -2c^{-2}\mu\chi_{AB}$, for $\underline{u}' \in [0, \underline{u}]$, we obtain

$$\frac{d}{d\underline{u}'}(|U_{\underline{u}'}|) = - \int_{U_{\underline{u}'}} c^{-2} \mu \text{tr} \chi d\mu_{\mathcal{G}}, \quad \frac{d}{d\underline{u}'}(|\partial U_{\underline{u}'}|) = - \int_{U_{\underline{u}'}} c^{-2} \mu \chi(\nu, \nu) ds,$$

where ν is the unit normal of $\partial U_{\underline{u}'}$ in $S_{t,0}$ and ds the element of arc length of $\partial U_{\underline{u}'}$. In view of the estimates on χ and μ derived before, for sufficiently small δ , we have

$$\frac{d}{d\underline{u}'}(|U_{\underline{u}'}|) \lesssim |U_{\underline{u}'}|, \quad \frac{d}{d\underline{u}'}(|\partial U_{\underline{u}'}|) \lesssim |\partial U_{\underline{u}'}|.$$

Therefore, by integrating from 0 to \underline{u} , we have

$$|U| \sim |U_{\underline{u}}|, \quad |\partial U| \sim |\partial U_{\underline{u}}|.$$

Hence, $I(t, \underline{u}) \sim I(t, 0) \sim 1$. Finally, since $|S_{t,\underline{u}}| \sim t^2$, we conclude that

$$(-t)^{3/2} \|f\|_{W^{1,4}(S_{t,\underline{u}})} + (-t) \|f\|_{L^\infty(S_{t,\underline{u}})} \lesssim \|f\|_{L^2(S_{t,\underline{u}})} + \|R_i f\|_{L^2(S_{t,\underline{u}})} + \|R_i R_j f\|_{L^2(S_{t,\underline{u}})}. \quad (3.42)$$

We remark that, similarly, we have

$$\|f\|_{L^4(S_{t,\underline{u}})} \lesssim \|f\|_{L^2(S_{t,\underline{u}})} + \|R_i f\|_{L^2(S_{t,\underline{u}})}. \quad (3.43)$$

As in [18], we also have the following elliptic estimates for traceless two-tensors.

Lemma 3.8. *If δ is sufficiently small, for any traceless 2-covariant symmetric tensor θ_{AB} on $S_{t,\underline{u}}$, we have*

$$\int_{S_{t,\underline{u}}} \mu^2 (|\nabla \theta|^2 + |\theta|^2) d\mu_{\mathcal{G}} \lesssim \int_{S_{t,\underline{u}}} \mu^2 |\mathcal{L} \theta|^2 + |\mathcal{L} \mu|^2 |\theta|^2 d\mu_{\mathcal{G}} \quad (3.44)$$

4. THE BEHAVIOR OF THE INVERSE DENSITY FUNCTION

As in [18], the behavior of the inverse density function μ also plays an dominant rôle in this paper. The method of obtaining estimates on μ is to relate μ to its initial value on $\Sigma_{t=-r_0}$. Besides the behavior with respect to δ , in this paper we will also take into account the behavior with respect to t . Since the metric g depends only on $\psi_0 = \partial_t \phi$, μ is also determined by ψ_0 . This leads naturally to the study of the wave equation $\square_{\tilde{g}} \psi_0 = 0$. We rewrite it in the null frame as

$$\underline{L}(L\psi_0) + \frac{1}{2} \text{tr} \tilde{\chi} \cdot L\psi_0 + \left(-\mu \Delta \psi_0 + \frac{1}{2} \text{tr} \tilde{\chi} \cdot \underline{L}\psi_0 + 2\zeta \cdot \mathcal{L}\psi_0 + \mu \mathcal{L} \log(c) \cdot \mathcal{L}\psi_0 \right) = 0. \quad (4.1)$$

4.1. The asymptotic expansion for μ .

Lemma 4.1. *For sufficiently small δ , we have*

$$|(-t)L\psi_0(t, \underline{u}, \theta) - r_0 L\psi_0(-r_0, \underline{u}, \theta)| \lesssim \delta^{\frac{1}{2}} M^3 (-t)^{-1}. \quad (4.2)$$

Proof. We regard (4.1) as a transport equation for $L\psi_0$. According to (B.1) and the estimates from previous sections, the L^∞ norm of the terms in the big parenthesis in (4.1) is bounded by $\delta^{\frac{1}{2}} M^3$. Hence,

$$|\underline{L}(L\psi_0)(t, \underline{u}, \theta) + \frac{1}{2} \text{tr} \tilde{\chi}(t, \underline{u}, \theta) \cdot L\psi_0(t, \underline{u}, \theta)| \lesssim \delta^{\frac{1}{2}} M^3 (-t)^{-3}.$$

By virtue of (3.24), this implies $|\underline{L}(L\psi_0)(t, \underline{u}, \theta) - \frac{1}{\underline{u}-t} L\psi_0(t, \underline{u}, \theta)| \lesssim \delta^{\frac{1}{2}} M^3 (-t)^{-3}$. Therefore, we obtain

$$|\underline{L}((\underline{u}-t)L\psi_0)(t, \underline{u}, \theta)| \lesssim \delta^{\frac{1}{2}} M^3 (-t)^{-2}.$$

Since $|\underline{u}| \leq \delta$, we integrate from $-r_0$ to t and this yields the desired estimates. \square

Remark 4.2. The estimates 4.2 also hold for $R_i L\psi_0$ or $R_i R_j \psi_0$, e.g., see (4.12). To derive these estimates, we commute R_i 's with (4.1) and follow the same way as in the above proof.

Since $L = c^{-2}\mu\underline{L} + 2T$, as a corollary, we have

Corollary 4.3. For sufficiently small δ , we have

$$|(-t)T\psi_0(t, \underline{u}, \theta) - r_0 T\psi_0(-r_0, \underline{u}, \theta)| \lesssim \delta^{\frac{1}{2}} M^3 (-t)^{-1}, \quad (4.3)$$

$$|(-t)\psi_0(t, \underline{u}, \theta) - r_0 \psi_0(-r_0, \underline{u}, \theta)| \lesssim \delta^{\frac{3}{2}} M^3 (-t)^{-1}. \quad (4.4)$$

We turn to the behavior of $\underline{L}\mu$.

Lemma 4.4. For sufficiently small δ , we have

$$|t^2 \underline{L}\mu(t, \underline{u}, \theta) - r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta)| \lesssim \delta M^4 (-t)^{-1}. \quad (4.5)$$

Proof. According to (2.34), we write $t^2 \underline{L}\mu(t, \underline{u}, \theta) - r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta)$ as

$$(t^2 m(t, \underline{u}, \theta) - r_0^2 m(-r_0, \underline{u}, \theta)) + [t^2 (\mu \cdot e)(t, \underline{u}, \theta) - r_0^2 (\mu \cdot e)(-r_0, \underline{u}, \theta)].$$

In view of (3.14), we bound the terms in the bracket by $\delta M^4 (-t)^{-1}$ up to a universal constant. Therefore,

$$t^2 \underline{L}\mu(t, \underline{u}, \theta) - r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) = (t^2 m(t, \underline{u}, \theta) - r_0^2 m(-r_0, \underline{u}, \theta)) + O\left(\frac{\delta M^4}{-t}\right).$$

Since

$$t^2 m(t, \underline{u}, \theta) - r_0^2 m(-r_0, \underline{u}, \theta) = \tau^2 \frac{3G''(0)(\psi_0(\tau, \underline{u}, \theta) \cdot T\psi_0(\tau, \underline{u}, \theta))}{(1 + 3G''(0)\psi_0^2(\tau, \underline{u}, \theta))^2} \Big|_{\tau=-r_0}^{\tau=t}.$$

It is clear that the estimates follow immediately after (4.3) and (4.4). \square

We now are able to prove an accurate estimate on μ .

Proposition 4.5. For sufficiently small δ , we have

$$\left| \mu(t, \underline{u}, \theta) - 1 + r_0^2 \left(\frac{1}{t} + \frac{1}{r_0} \right) \underline{L}\mu(-r_0, \underline{u}, \theta) \right| \lesssim \delta M^4 (-t)^{-2}. \quad (4.6)$$

In particular, we have $\mu \leq C_0$ where C_0 is a universal constant depending only on the initial data.

Proof. According to the previous lemma, we integrate $\underline{L}\mu$:

$$\begin{aligned} \mu(t, \underline{u}, \theta) - \mu(-r_0, \underline{u}, \theta) &= \int_{-r_0}^t \underline{L}\mu(\tau, \underline{u}, \theta) d\tau = \int_{-r_0}^t \frac{\tau^2 \underline{L}\mu(\tau, \underline{u}, \theta)}{\tau^2} d\tau \\ &\stackrel{(4.5)}{=} \int_{-r_0}^t \frac{r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta)}{\tau^2} + \frac{O(\delta M^4)}{-\tau^3} d\tau \\ &= -\left(\frac{1}{t} + \frac{1}{r_0}\right) r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) + O\left(\frac{\delta M^4}{t^2}\right). \end{aligned}$$

Therefore, we can use (3.4) to conclude. \square

We are ready to derive two key properties of the inverse density function μ . The first asserts that the shock wave region is trapping for μ .

Proposition 4.6. For sufficiently small δ and for all $(t, \underline{u}, \theta) \in W_{\text{shock}}$, we have

$$\underline{L}\mu(t, \underline{u}, \theta) \leq -\frac{1}{4|t|^2}. \quad (4.7)$$

Proof. For $(t, \underline{u}, \theta) \in W_{shock}$, we have $\mu(t, \underline{u}, \theta) < \frac{1}{10}$. In view of (4.6), we claim that $r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) < 0$. Otherwise, since $\frac{1}{t} + \frac{1}{r_0} < 0$, we would have $\mu(t, \underline{u}, \theta) \geq 1 + O(\delta M^2) > \frac{1}{10}$, provided δ is sufficiently small. This contradicts the fact that $\mu(t, \underline{u}, \theta) < \frac{1}{10}$.

We can also use this argument to show that $(\frac{1}{t} + \frac{1}{r_0})r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) \geq \frac{1}{2}$. Otherwise, for sufficiently small δ , we would have $\mu(t, \underline{u}, \theta) \geq \frac{1}{2} + O(\delta M^2) > \frac{1}{10}$.

Therefore, we obtain $r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) \leq \frac{1}{2} \frac{r_0 t}{r_0 + t}$. In view of (4.5), we have

$$t^2 \underline{L}\mu(t, \underline{u}, \theta) \leq \frac{1}{2} \frac{r_0 t}{r_0 + t} + O\left(\frac{\delta M^2}{-t}\right).$$

By taking a sufficiently small δ and noticing that $\frac{r_0 t}{r_0 + t}$ is bounded from above by a negative number, this yields the desired estimates. \square

Remark 4.7. As in [18], compared to the estimates (3.15) $|\underline{L}\mu| \lesssim M^2(-t)^{-2}$, (4.5) $|\underline{L}\mu| \leq C_0(-t)^{-2} + \delta M^2(-t)^{-3}$, where C_0 depends only on the initial data, gives us a more precise estimate. The improvement comes from integrating the wave equation $\square_{\tilde{g}}\psi_0 = 0$ or equivalently (4.1).

4.2. The asymptotic expansion for derivatives of μ . We start with an estimate on derivatives of $\text{tr}\tilde{\chi}$.

Lemma 4.8. For sufficiently small $\delta \leq \varepsilon$, we have

$$\|L \text{tr}\tilde{\chi}\|_{L^\infty} \lesssim M^2(-t)^{-1}, \quad (4.8)$$

$$\|\not{d}\text{tr}\tilde{\chi}\|_{L^\infty} \lesssim \delta M^2(-t)^{-4}. \quad (4.9)$$

Proof. We derive a transport equation for $R_i \underline{\chi}'_{AB}$ by commuting R_i with (3.20):

$$\underline{L}(R_i \underline{\chi}'_{AB}) = [\underline{L}, R_i] \underline{\chi}'_{AB} + e R_i \underline{\chi}'_{AB} + 2 \underline{\chi}'_A{}^C R_i \underline{\chi}'_{BC} + (R_i e) \cdot \underline{\chi}'_{AB} - \frac{e}{\underline{u} - t} R_i \not{g}_{AB} - R_i \left(\frac{e}{\underline{u} - t} \right) \not{g}_{AB} - R_i \underline{\alpha}'_{AB}.$$

Since $[\underline{L}, R_i]^A = (R_i) \pi_{\underline{L}}^A$, the commutator term $[\underline{L}, R_i] \underline{\chi}'_{AB}$ can be bounded by the estimates on the deformation tensors. We then multiply both sides by $R_i \underline{\chi}'_{AB}$ and repeat the procedure that we used to derive (3.19). Since it is routine, we omit the details and only give the final result

$$\|R_i \left(\underline{\chi}_{AB} + \frac{\not{g}_{AB}}{\underline{u} - t} \right)\|_{L^\infty} \lesssim \delta M^2(-t)^{-3}. \quad (4.10)$$

In particular, this yields $\|R_i \text{tr}\tilde{\chi}\|_{L^\infty} \lesssim \delta M^2(-t)^{-3}$ which is equivalent to (4.9).

We derive a transport equation for $L \underline{\chi}'_{AB}$ by commuting L with (3.20):

$$\begin{aligned} \underline{L}(L \underline{\chi}'_{AB}) &= e L \underline{\chi}'_{AB} + 2 \underline{\chi}'_A{}^C L \underline{\chi}'_{BC} + L e \underline{\chi}'_{AB} - \frac{e}{\underline{u} - t} L \not{g}_{AB} \\ &\quad + L \left(\frac{e}{\underline{u} - t} \right) \not{g}_{AB} - L \underline{\alpha}'_{AB} + \underline{L}(c^{-2} \mu) \underline{L} \underline{\chi}'_{AB} + \not{g}^{CD} (\zeta_C + \eta_D) X_C (\underline{\chi}'_{AB}). \end{aligned}$$

We then use Gronwall to derive

$$\|L \left(\underline{\chi}_{AB} + \frac{\not{g}_{AB}}{\underline{u} - t} \right)\|_{L^\infty} \lesssim M^2(-t)^{-3}. \quad (4.11)$$

In particular, this yields $\|L \text{tr}\tilde{\chi}\|_{L^\infty} \lesssim M^2(-t)^{-1}$. This is equivalent to (4.8). \square

We now derive estimates for $R_i \psi_0$.

Lemma 4.9. For sufficiently small δ , we have

$$|(-t) L R_i \psi_0(t, \underline{u}, \theta) - r_0 L R_i \psi_0(-r_0, \underline{u}, \theta)| \lesssim \delta^{\frac{1}{2}} M^3(-t)^{-1}. \quad (4.12)$$

Proof. We commute R_i with (4.1) and we obtain that $\underline{L}(R_i L\psi_0) + \frac{1}{2}\text{tr}\tilde{\chi} \cdot R_i L\psi_0 = N$ with

$$N = R_i \left(\mu \underline{\Delta} \psi_0 - \frac{1}{2} \text{tr}\tilde{\chi} \cdot \underline{L}\psi_0 - 2\underline{\zeta} \cdot \not{d}\psi_0 - \mu \not{d} \log(c) \cdot \not{d}\psi_0 \right) - \frac{1}{2} R_i \text{tr}\tilde{\chi} \cdot L\psi_0 + [\underline{L}, R_i] L\psi_0.$$

According to (B.1) and the previous lemma, N is bounded by $\delta^{\frac{1}{2}} M^3$. Hence, $|\underline{L}(R_i L\psi_0) + \frac{1}{2}\text{tr}\tilde{\chi} \cdot R_i L\psi_0| \lesssim \delta^{\frac{1}{2}} M^3 (-t)^{-3}$. We then integrate to derive

$$|t| R_i L\psi_0(t, \underline{u}, \theta) - r_0 R_i L\psi_0(-r_0, \underline{u}, \theta) \lesssim \delta^{\frac{1}{2}} M^3 (-t)^{-1}.$$

The commutator $[R_i, \underline{L}]\psi_0$ is bounded by $\delta M^3 (-t)^{-2}$ thanks to the estimates on deformation tensors. This completes the proof. \square

Using this lemma, we can obtain a more accurate estimate for $\not{d}\mu$.

Lemma 4.10. *For sufficiently small δ , we have*

$$\|\not{d}\mu\|_{L^\infty(\Sigma_t)} \lesssim \frac{(1 + \delta M^4)}{t^2}. \quad (4.13)$$

Proof. We commute R_i with $\underline{L}\mu = m + e\mu$ to derive

$$\underline{L}(R_i \mu) = R_i m + (e R_i \mu + \mu R_i e + [\underline{L}, R_i] \mu).$$

According to (B.1) and the estimates on $L R_i \psi_0$ (needed to bound $R_i m$) from the previous lemma, it is straightforward to bound the terms in the parenthesis by δM^2 . Similar to (4.5), we obtain

$$|(-t)^2 \underline{L}(R_i \mu)(t, \underline{u}, \theta) - r_0^2 \underline{L}(R_i \mu)(-r_0, \underline{u}, \theta)| \lesssim \delta M^4 (-t)^{-1}. \quad (4.14)$$

Since $\|[\underline{L}, R_i]\|_{L^\infty} \lesssim \delta M^2 (-t)^{-2}$, we bound $R_i \mu$ as

$$\begin{aligned} R_i \mu(t, \underline{u}, \theta) - R_i \mu(-r_0, \underline{u}, \theta) &\stackrel{(4.14)}{=} \int_{-r_0}^t \frac{r_0^2 \underline{L}(R_i \mu(r_0, \underline{u}, \theta))}{\tau^2} + \frac{O(\delta M^4)}{-\tau^3} d\tau \\ &= -\left(\frac{1}{t} + \frac{1}{r_0}\right) r_0^2 \underline{L}(R_i \mu(-r_0, \underline{u}, \theta)) + \frac{O(\delta M^4)}{t^2}. \end{aligned}$$

By using the relation between R_i and \not{d} , this inequality yields (4.13) for sufficiently small δ . \square

We can also obtain a better estimate for $L\mu$.

Lemma 4.11. *For sufficiently small δ , we have*

$$\|L\mu\|_{L^\infty(\Sigma_t)} \lesssim \frac{\delta^{-1} + M^4}{-t}. \quad (4.15)$$

Proof. By commuting L with $\underline{L}\mu = m + e\mu$, we obtain

$$\underline{L}(L\mu) = Lm + [-2(\underline{\zeta}^A + \underline{\eta}^A)X_A(\mu) + \underline{L}(c^{-2}\mu)\underline{L}\mu + eL\mu + \mu Le].$$

According to (B.1), we can bound the terms in the bracket by $M^4 (-t)^{-3}$. Hence,

$$|t|^2 \underline{L}(L\mu)(t, \underline{u}, \theta) - |r_0|^2 \underline{L}(L\mu)(-r_0, \underline{u}, \theta) = |t|^2 Lm(t, \underline{u}, \theta) - |r_0|^2 Lm(-r_0, \underline{u}, \theta) + \frac{O(M^4)}{-t}.$$

By the explicit formula of m , we can proceed exactly as in Lemma 4.4 and we obtain

$$|t|^2 \underline{L}(L\mu)(t, \underline{u}, \theta) - |r_0|^2 \underline{L}(L\mu)(-r_0, \underline{u}, \theta) \lesssim M^4 (-t)^{-1}. \quad (4.16)$$

We then integrate along \underline{L} and we have

$$\begin{aligned} L\mu(t, \underline{u}, \theta) - L\mu(-r_0, \underline{u}, \theta) &= \int_{-r_0}^t \frac{\tau^2 \underline{L}(L\mu(\tau, \underline{u}, \theta))}{\tau^2} d\tau \stackrel{(4.16)}{=} \int_{-r_0}^t \frac{r_0^2 \underline{L}(L\mu(r_0, \underline{u}, \theta))}{\tau^2} + \frac{O(M^4)}{-\tau^3} d\tau \\ &= -\left(\frac{1}{t} + \frac{1}{r_0}\right) r_0^2 \underline{L}(L\mu(-r_0, \underline{u}, \theta)) + \frac{O(M^4)}{t^2}. \end{aligned}$$

For sufficiently small δ , this implies (4.15). \square

We now relate $L^2\psi_0(t, \underline{u}, \theta)$ to its initial value.

Lemma 4.12. *For sufficiently small δ , we have*

$$|t|L^2\psi_0(t, \underline{u}, \theta) - r_0L^2\psi_0(-r_0, \underline{u}, \theta)| \lesssim \delta^{-\frac{1}{2}}M^3(-t)^{-1}. \quad (4.17)$$

Proof. We commute L with (4.1) and we obtain the following transport equation for $L^2\psi_0$:

$$\begin{aligned} \underline{L}(L^2\psi_0) + \frac{1}{2}\text{tr}\tilde{\chi} \cdot L^2\psi_0 &= -\frac{1}{2}L(\text{tr}\tilde{\chi}) \cdot L\psi_0 + [\underline{L}, L]L\psi_0 \\ &\quad + L\left(\mu\Delta\psi_0 - \frac{1}{2}\text{tr}\tilde{\chi} \cdot \underline{L}\psi_0 - 2\underline{\zeta} \cdot \not{d}\psi_0 - \mu\not{d}\log(c) \cdot \not{d}\psi_0\right). \end{aligned} \quad (4.18)$$

The righthand side of the above equation can be expanded as

$$\begin{aligned} &L\text{tr}\tilde{\chi}L\psi_0 + \underline{L}(c^{-2}\mu)\underline{L}L\psi_0 + (\underline{\eta} + \underline{\zeta})\not{d}L\psi_0 + \left(L\mu\Delta\psi_0 + \mu L\Delta\psi_0 + L\text{tr}\tilde{\chi}\underline{L}\psi_0 + \text{tr}\tilde{\chi}\underline{L}L\psi_0\right. \\ &\quad \left.+ L\underline{\zeta}\not{d}\psi_0 + \underline{\zeta}L\not{d}\psi_0 + L\mu \cdot \not{d}\log(c) \cdot \not{d}\psi_0 + \mu L(\not{d}\log(c)) \cdot \not{d}\psi_0 + \mu\not{d}\log(c) \cdot L\not{d}\psi_0\right). \end{aligned}$$

Since the exact numeric constants and signs for the coefficients are irrelevant for estimates, we replace all of them by 1 in the above expressions.

Since $\underline{\zeta}_A = -c^{-1}\mu X_A(c)$, by applying L and using (4.15), we obtain $|L\underline{\zeta}| \lesssim M^2$. Therefore, according (4.8), (B.1) and the estimates derived previously in this section, we can bound all the terms on the right hand side and we obtain

$$|\underline{L}(L^2\psi_0) + \frac{1}{2}\text{tr}\tilde{\chi} \cdot L^2\psi_0| \lesssim \delta^{-\frac{1}{2}}M^3(-t)^{-3}.$$

We then integrate from $-r_0$ to t to obtain (4.17). \square

Following the same procedure, we have:

Lemma 4.13. *For sufficiently small δ , we have*

$$\|L^2\text{tr}\tilde{\chi}\| \lesssim M^2\delta^{-1}(-t)^{-1}, \quad (4.19)$$

$$|t|L^3\psi_0(t, \underline{u}, \theta) - r_0L^3\psi_0(-r_0, \underline{u}, \theta)| \lesssim \delta^{-\frac{3}{2}}M^3(-t)^{-1}. \quad (4.20)$$

We omit the proof since it is routine. Similarly, we commute L twice with $\underline{L}\mu = m + e\mu$, we can use (4.19) and (4.20) to obtain

Lemma 4.14. *There exists $\varepsilon = \varepsilon(M)$ so that for all $\delta \leq \varepsilon$, we have*

$$\|L^2\mu\|_{L^\infty(\Sigma_t)} \lesssim (\delta^{-2} + \delta^{-1}M^2)(-t)^{-1}, \quad (4.21)$$

$$\|T^2\mu\|_{L^\infty(\Sigma_t)} \lesssim (\delta^{-2} + \delta^{-1}M^2)(-t)^{-1}. \quad (4.22)$$

We turn to the improved estimate for $\mu^{-1}T\mu$.

Proposition 4.15. *Let s be such that $t < s < t^*$. For $p = (t, \underline{u}, \theta) \in W_\delta$, let $(\mu^{-1}T\mu)_+$ be the nonnegative part of $\mu^{-1}T\mu$. For sufficiently small δ and for all $p \in W_{shock}$, we have*

$$(\mu^{-1}T\mu)_+(t, \underline{u}, \theta) \lesssim \frac{1}{|t-s|^{\frac{1}{2}}} \delta^{-1} (-t)^{1/2}. \quad (4.23)$$

Proof. As in [18], by a maximum principle argument, we have:

$$\|\mu^{-1}(T\mu)_+\|_{L^\infty([0, \delta])} \leq \sqrt{\frac{\|T^2\mu\|_{L^\infty([0, \delta])}}{\inf_{\underline{u} \in [0, \delta]} \mu(\underline{u})}}. \quad (4.24)$$

From the previous lemma, we have:

$$\|T^2\mu\|_{L^\infty([- \delta, \delta])} \lesssim \frac{1}{|t|} \delta^{-2}. \quad (4.25)$$

For $\inf \mu$, we assume $(t, \underline{u}, \theta) \in W_{shock}$. According to (4.7), the condition $(t, \underline{u}, \theta) \in W_{shock}$ implies $|t|^2(L\mu)(t, \underline{u}, \theta) \leq -\frac{1}{4}$. Therefore we have

$$\begin{aligned} \mu(t, \underline{u}, \theta) &= \mu(s, \underline{u}, \theta) - \int_t^s \underline{L}\mu(\tau, \underline{u}, \theta) \geq - \int_t^s \underline{L}\mu(\tau, \underline{u}, \theta) \\ &\geq \int_t^s \frac{1}{4\tau^2} d\tau \geq \frac{s-t}{4t^2}. \end{aligned}$$

Together with (4.24) and (4.25), this completes the proof. \square

5. ENERGY ESTIMATES FOR LINEAR EQUATION

In this section we establish the energy estimates for the following inhomogeneous equation

$$\square_{\tilde{g}} \psi = \rho \quad (5.1)$$

Since \tilde{g} depends on ϕ , we will need to handle the error terms contributed by the deformation tensors of the multiplier vectorfields. In addition, since we need to deal with the time decay, we have to modify the multipliers K_0 and K_1 in [18], which will be specified later in this section.

5.1. Energy and Flux. As usual, we introduce the energy-momentum tensor, which is the same with respect to g and \tilde{g} :

$$T_{\mu\nu}(\psi) := \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi, \quad (5.2)$$

and we have the decomposition for $T_{\mu\nu}$ with respect to the null frame $(L, \underline{L}, X_1, X_2)$:

$$\begin{aligned} T_{LL} &= (L\psi)^2, \quad T_{\underline{L}\underline{L}} = (\underline{L}\psi)^2, \quad T_{\underline{L}L} = \mu |\not{d}\psi|^2, \quad T_{LA} = L\psi \cdot X_A(\psi), \\ T_{\underline{L}A} &= \underline{L}\psi \cdot X_A(\psi), \quad T_{AB} = X_A(\psi) X_B(\psi) - \frac{1}{2} \not{g}_{AB} (-\mu^{-1} L\psi \underline{L}\psi + |\not{d}\psi|^2). \end{aligned} \quad (5.3)$$

We use two *multiplier vectorfields* $K_0 = L + \underline{L}$ and $K_1 = \left(\frac{2(-t)}{\widetilde{\text{tr}} \chi} \right) \underline{L}$. Following a similar argument as in [18], the associated energy and flux for K_0 are given by:

$$E^0(t, \underline{u}) = \int_{\Sigma_t^{\underline{u}}} \frac{1}{2c} ((L\psi)^2 + c^{-2} \mu (\underline{L}\psi)^2 + (\mu + c^{-2} \mu^2) |\not{d}\psi|^2), \quad F^0(t, \underline{u}) = \int_{\underline{C}_{\underline{u}}^t} \frac{1}{c} ((\underline{L}\psi)^2 + \mu |\not{d}\psi|^2), \quad (5.4)$$

which satisfy

$$E^0(t, \underline{u}) \sim E(\psi)(t, \underline{u}), \quad F^0(t, \underline{u}) \sim F(\psi)(t, \underline{u})$$

The energy estimates will be based on the following identity:

$$E^0(t, \underline{u}) - E^0(-r_0, \underline{u}) + F^0(t, \underline{u}) = \int_{W_t^{\underline{u}}} c^{-2} \widetilde{Q}_0 \quad (5.5)$$

where

$$\widetilde{Q}_0 := -\rho \cdot K_0 \psi - \frac{1}{2} \widetilde{T}^{\mu\nu} \widetilde{\pi}_{0,\mu\nu},$$

with $\widetilde{\pi}_{0,\mu\nu} = (\mathcal{L}_{K_0} \widetilde{g})_{\mu\nu}$ being the deformation tensor of K_0 . In the above identity, the spacetime integral is defined as follows:

$$\int_{W_t^{\underline{u}}} f = \int_{-r_0}^t \int_0^{\underline{u}} \left(\int_{S_{\tau, \underline{u}'}} \mu \cdot f(\tau, \underline{u}', \theta) d\mu_g \right) d\underline{u}' d\tau. \quad (5.6)$$

The discussion for K_1 is much more complicated. First, a standard calculation as in [18] implies the following energy flux associated to K_1 :

$$\begin{aligned} E^1(t, \underline{u}) &= \int_{\Sigma_t^{\underline{u}}} \frac{1}{2c} \left(\frac{2(-t)}{\widetilde{\text{tr}} \underline{\chi}} (c^{-2} \mu(\underline{L}\psi)^2 + \mu|\not{d}\psi|^2) + t\psi (c^{-2} \mu(\underline{L}\psi) + \underline{L}\psi) + 2c^{-2} \mu\psi^2 \right) \\ F^1(t, \underline{u}) &= \int_{\underline{C}_t^{\underline{u}}} \frac{1}{c} \left(\frac{2(-t')}{\widetilde{\text{tr}} \underline{\chi}} (\underline{L}\psi)^2 + t\psi(\underline{L}\psi) + \frac{1}{2} \psi^2 \right) \end{aligned} \quad (5.7)$$

Again, the energy estimates will be based on the following identity:

$$E^1(t, \underline{u}) - E^1(-r_0, \underline{u}) + F^1(t, \underline{u}) = \int_{W_t^{\underline{u}}} c^{-2} \widetilde{Q}_1 \quad (5.8)$$

where

$$\widetilde{Q}_1 := -\rho \cdot (K_1 \psi - t\psi) - \frac{1}{2} \widetilde{T}^{\mu\nu} \widetilde{\pi}_{1,\mu\nu} + t\widetilde{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} \psi^2 \square_{\widetilde{g}}(t),$$

which can be written as

$$\widetilde{Q}_1 := -\rho \cdot (K_1 \psi - t\psi) - \frac{1}{2} \widetilde{T}^{\mu\nu} \widetilde{\pi}'_{1,\mu\nu} - \frac{1}{2} \psi^2 \square_{\widetilde{g}}(t),$$

with $\widetilde{\pi}_{1,\mu\nu} = (\mathcal{L}_{K_1} \widetilde{g})_{\mu\nu}$ being the deformation tensor of K_1 and $\widetilde{\pi}'_{1,\mu\nu} = \widetilde{\pi}_{1,\mu\nu} + 2t\widetilde{g}_{\mu\nu}$. However, unlike the case of K_0 , it is not straightforward to show that $E^1(t, \underline{u}), F^1(t, \underline{u})$ are equivalent to $\underline{E}(\psi)(t, \underline{u}), \underline{F}(\psi)(t, \underline{u})$. So instead of $E^1(t, \underline{u}), F^1(t, \underline{u})$, we work with

$$E'^1(t, \underline{u}) = \int_{\Sigma_t^{\underline{u}}} \frac{1}{2c} \frac{4(-t)}{\widetilde{\text{tr}} \underline{\chi}} \left(c^{-2} \mu \left(\underline{L}\psi + \frac{1}{2} \widetilde{\text{tr}} \underline{\chi} \psi \right)^2 + \mu|\not{d}\psi|^2 \right), \quad F'^1(t, \underline{u}) = \int_{\underline{C}_t^{\underline{u}}} \frac{1}{c} \frac{4(-t')}{\widetilde{\text{tr}} \underline{\chi}} \left(\underline{L}\psi + \frac{1}{2} \widetilde{\text{tr}} \underline{\chi} \psi \right)^2 \quad (5.9)$$

Now it is straightforward to show

$$E'^1(t, \underline{u}) \sim \underline{E}(\psi)(t, \underline{u}), \quad F'^1(t, \underline{u}) \sim \underline{F}(\psi)(t, \underline{u})$$

So we need to establish an identity for $E'^1(t, \underline{u})$ and $F'^1(t, \underline{u})$ similar to (5.8). A direct calculation implies

$$E^1(t, \underline{u}) - E'^1(t, \underline{u}) = \int_{\Sigma_t^{\underline{u}}} \frac{1}{2c} \left(2(-t)\psi(T\psi) + \left(c^{-2} \mu + \frac{1}{2} c^{-2} \mu t \widetilde{\text{tr}} \underline{\chi} \right) \psi^2 \right) \quad (5.10)$$

By using the relations

$$\not{L}_T d\mu_g = c^{-1} \mu \text{tr} \theta d\mu_g, \quad \text{tr} \underline{\chi} = -c \text{tr} \theta, \quad \text{tr} \chi = c^{-1} \mu \text{tr} \theta,$$

the difference can be rewritten as:

$$E^1(t, \underline{u}) - E'^1(t, \underline{u}) = \int_{S_{t, \underline{u}}} \frac{1}{2c} (-t) \psi^2 - I \quad (5.11)$$

where

$$I = \int_{\Sigma_t^{\underline{u}}} \frac{1}{2c} \left(\frac{1}{2} \widetilde{\text{tr}} \chi(-t) - c^{-2} \mu \right) \psi^2 \quad (5.12)$$

We also compute the difference between $F^1(t, \underline{u})$ and $F'^1(t, \underline{u})$:

$$F^1(t, \underline{u}) - F'^1(t, \underline{u}) = - \int_{\underline{C}_{\underline{u}}^t} \frac{1}{2c} \left(\underline{L}((-t) \psi^2) + \widetilde{\text{tr}} \underline{\chi}(-t) \psi^2 \right) \quad (5.13)$$

Using the identity:

$$\not{L} \underline{L} d\mu_g = \text{tr} \underline{\chi} d\mu_g$$

This difference is rewritten as:

$$F^1(t, \underline{u}) - F'^1(t, \underline{u}) = - \int_{S_{t, \underline{u}}} \frac{1}{2c} (-t) \psi^2 + \int_{S_{-r_0, \underline{u}}} \frac{1}{2c} r_0 \psi^2 \quad (5.14)$$

Substituting (5.11) and (5.14) into (5.8), we have:

$$\begin{aligned} & E'^1(t, \underline{u}) + F'^1(t, \underline{u}) + \int_{\Sigma_t^{\underline{u}}} \frac{1}{2c} \left(\frac{1}{2} \widetilde{\text{tr}} \chi(-t) - c^{-2} \mu \right) \psi^2 \\ &= E'^1(-r_0, \underline{u}) + \int_{\Sigma_{-r_0}^{\underline{u}}} \frac{1}{2c} \left(\frac{1}{2} \widetilde{\text{tr}} \chi(-t) - c^{-2} \mu \right) \psi^2 + \int_{W_t^{\underline{u}}} c^{-2} \widetilde{Q}_1 \end{aligned} \quad (5.15)$$

To estimate the term involving ψ^2 , one needs the following lemma:

Lemma 5.1. *For a ψ which vanishes on \underline{C}_0 , we have*

$$\int_{S_{t, \underline{u}}} \psi^2 \lesssim \delta \int_{\Sigma_t^{\underline{u}}} (L\psi)^2 + \mu(\underline{L}\psi)^2 \lesssim \delta E(\psi)(t, \underline{u}), \quad \int_{\Sigma_t^{\underline{u}}} \psi^2 \lesssim \delta^2 E(\psi)(t, \underline{u}) \quad (5.16)$$

The goal of this section is to bound $E(t, \underline{u})$, $F(t, \underline{u})$ and $\underline{E}(t, \underline{u})$ and $\underline{F}(t, \underline{u})$ in terms of their corresponding initial data and ρ .

Proposition 5.2. *The limits $\lim_{r_0 \rightarrow \infty} E^0(-r_0, \underline{u})$ and $\lim_{r_0 \rightarrow \infty} E'^1(-r_0, \underline{u})$ both exist and we have*

$$\lim_{r_0 \rightarrow \infty} E^0(-r_0, \underline{u}) \lesssim 1, \quad \lim_{r_0 \rightarrow \infty} E'^1(-r_0, \underline{u}) \lesssim \delta^2. \quad (5.17)$$

Proof. For the proof we choose $\underline{u} = \delta$ and the proof for any $\underline{u} \in [0, \delta]$ is similar. In view of the definitions of $E^0(-r_0, \underline{u})$ and $E'^1(-r_0, \underline{u})$ and the fact that

$$\mu|_{t=-r_0} = c, \quad \widetilde{\text{tr}} \underline{\chi}|_{t=-r_0} = 2\underline{L} \left(\frac{1}{c} \right) - \frac{2}{r_0},$$

it suffices to prove that the limits $\lim_{r_0 \rightarrow \infty} \|T\psi\|_{L^2(\Sigma_{-r_0}^\delta)}$, $\lim_{r_0 \rightarrow \infty} \|r_0 \underline{L}\psi\|_{L^2(\Sigma_{-r_0}^\delta)}$, $\lim_{r_0 \rightarrow \infty} \|r_0 \not{L}\psi\|_{L^2(\Sigma_{-r_0}^\delta)}$, $\lim_{r_0 \rightarrow \infty} \|r_0 \psi\|_{L^2(\Sigma_{-r_0}^\delta)}$ exist and satisfy

$$\lim_{r_0 \rightarrow \infty} \|T\psi\|_{L^2(\Sigma_{-r_0}^\delta)} \lesssim 1, \quad \lim_{r_0 \rightarrow \infty} \|r_0 \underline{L}\psi\|_{L^2(\Sigma_{-r_0}^\delta)}, \quad \lim_{r_0 \rightarrow \infty} \|r_0 \not{L}\psi\|_{L^2(\Sigma_{-r_0}^\delta)}, \quad \lim_{r_0 \rightarrow \infty} \|r_0 \psi\|_{L^2(\Sigma_{-r_0}^\delta)} \lesssim \delta.$$

Here we only give a detailed proof for $\lim_{r_0 \rightarrow \infty} \|T\psi\|_{L^2(\Sigma_{-r_0}^\delta)}$ and for $\psi = \partial_t \phi$. The proof for $\psi = \partial_i \phi$ and for \underline{L}, \not{L} derivatives are similar. According to the initial data constructed in Lemma 1.1, we have

$$T\psi(-r_0, \theta) = \frac{\delta^{-1/2}}{r_0} (\partial_s \phi_1) \left(\frac{r - r_0}{\delta}, \theta \right)$$

where ∂_s is derivative of $\phi_1(s, \theta)$ with respect to its first argument. A direct computation shows

$$\begin{aligned} \lim_{r_0 \rightarrow \infty} \|T\psi\|_{L^2(\Sigma_{-r_0}^\delta)}^2 &= \lim_{r_0 \rightarrow \infty} \int_{r_0}^{r_0+\delta} \int_{\mathbb{S}^2} \frac{\delta^{-1} r^2}{r_0^2} (\partial_s \phi_1)^2 \left(\frac{r - r_0}{\delta}, \theta \right) d\mu_{\mathbb{S}^2} dr \\ &= \lim_{r_0 \rightarrow \infty} \int_0^1 \int_{\mathbb{S}^2} \frac{(r_0 + \delta s)^2}{r_0^2} (\partial_s \phi_1)^2(s, \theta) d\mu_{\mathbb{S}^2} ds \\ &= \int_0^1 \int_{\mathbb{S}^2} (\partial_s \phi_1)^2(s, \theta) d\mu_{\mathbb{S}^2} ds. \end{aligned}$$

Applying the argument in the proof of Lemma 1.1 to $\partial_t \phi$ instead of ϕ , one can see that $\underline{L}\psi$ can also be written as

$$(\underline{L}\psi)(-r_0, \theta) = \frac{\delta^{1/2}}{r_0^2} \phi_3 \left(\frac{r - r_0}{\delta}, \theta \right)$$

for some smooth function $\phi_3(s, \theta)$ which vanishes for $s \leq 0$. Therefore the above argument applies to $\underline{L}\psi, \not{L}\psi$ and ψ . \square

Remark 5.3. From the above proof, one can see that Proposition 5.2 is also valid if the commutators $R_i, T, t\underline{L}$ are applied on $\psi := \partial_t \phi, \partial_i \phi$. Therefore the higher order initial energies are also finite at $t = -\infty$, and our energy estimates independent of r_0 implies the existence of semi-global-in-time solutions which lead to shock formation.

5.2. Error terms. Now we study the error terms \tilde{Q}_0 and \tilde{Q}_1 . The deformation tensor $\tilde{\pi}_{0,\mu\nu}$ is given by:

$$\begin{aligned} \tilde{\pi}_{0,LL} &= \frac{4}{c} \mu \underline{L}(c^{-2} \mu), \quad \tilde{\pi}_{0,\underline{L}\underline{L}} = 0 \\ \tilde{\pi}_{0,L\underline{L}} &= -\frac{2}{c} \mu (\mu^{-1}(K_0 \mu) - (K_0 \log c) + 2\underline{L}(c^{-2} \mu)) \\ \tilde{\pi}_{0,LA} &= \frac{2}{c} \left((\zeta_A + \eta_A) - \mu \nabla_A(c^{-2} \mu) \right) \\ \tilde{\pi}_{0,\underline{L}A} &= -\frac{2}{c} (\zeta_A + \eta_A) \\ \hat{\not{\pi}}_{0,AB} &= \frac{2}{c} (\hat{\chi}_{AB} + \hat{\chi}_{AB}) \\ \text{tr} \tilde{\not{\pi}}_0 &= 2 \left(\text{tr} \tilde{\chi} + \text{tr} \tilde{\chi} \right) \end{aligned}$$

The modified deformation tensor $\tilde{\pi}'_{1,\mu\nu}$ is given by:

$$\begin{aligned}\tilde{\pi}'_{1,LL} &= \frac{4}{c}\mu \left(\left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) \underline{L}(c^{-2}\mu) - L \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) \right), \quad \tilde{\pi}'_{1,\underline{L}\underline{L}} = 0 \\ \tilde{\pi}'_{1,L\underline{L}} &= -\frac{2}{c}\mu \left(\mu^{-1}K_1\mu + \underline{L} \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) + 2t - K_1 \log c \right) \\ \tilde{\pi}'_{1,LA} &= \frac{2}{c} \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) (\zeta_A + \eta_A) - \mu \nabla_A \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right), \quad \tilde{\pi}'_{1,\underline{L}A} = 0 \\ \hat{\pi}'_{1,AB} &= \frac{2}{c} \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) \hat{\chi}_{AB}, \quad \text{tr} \tilde{\pi}'_1 = 0\end{aligned}$$

Remark 5.4. As we stated in the introduction, the choice of $K_1 = \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) \underline{L}$ is such that it behaves like $u^2 \underline{L}$ (up to a multiplication by a constant) when $|t|$ is large. On the other hand, the specific choice of the coefficient $\left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right)$ is to guarantee that $\text{tr} \tilde{\pi}'_1$ vanishes, which would cause a divergence in time if it is non-zero.

To calculate \tilde{Q}_0 and \tilde{Q}_1 , we need to raise the indices for $T_{\mu\nu}$:

$$\begin{aligned}T^{LL} &= \frac{(L\psi)^2}{4\mu^2}, \quad T^{\underline{L}\underline{L}} = \frac{(L\psi)^2}{4\mu^2}, \quad T^{\underline{L}L} = \frac{(\not{L}\psi)^2}{4\mu}, \quad T^{LA} = -\frac{\underline{L}\psi X_A(\psi)}{2\mu}, \\ T^{\underline{L}A} &= -\frac{L\psi X_A(\psi)}{2\mu}, \quad T^{AB} = \not{g}^{AC} \not{g}^{BD} X_C(\psi) X_D(\psi) - \frac{1}{2} \not{g}^{AB} \left(-\frac{L\psi \underline{L}\psi}{\mu} + |\not{L}\psi|^2 \right).\end{aligned}\tag{5.18}$$

Now we can compute the integrands \tilde{Q}_0 and \tilde{Q}_1 explicitly. For \tilde{Q}_0 , we have

$$\begin{aligned}c^{-2}\tilde{Q}_0 &= -c^{-2}\rho \cdot K_0\psi - \frac{1}{2}T^{\mu\nu}\tilde{\pi}_{0,\mu\nu} = \tilde{Q}_{0,0} + \tilde{Q}_{0,1} + \tilde{Q}_{0,2} + \tilde{Q}_{0,3} + \tilde{Q}_{0,4} + \tilde{Q}_{0,5} \\ &= -c^{-2}\rho \cdot K_0\psi - T^{\underline{L}\underline{L}}\tilde{\pi}_{0,\underline{L}\underline{L}} - T^{LA}\tilde{\pi}_{0,LA} - T^{\underline{L}A}\tilde{\pi}_{0,\underline{L}A} - \frac{1}{2}T^{AB}\tilde{\pi}_{0,AB} + \frac{1}{2}T^{LL}\tilde{\pi}_{0,LL}.\end{aligned}\tag{5.19}$$

The $\tilde{Q}_{0,i}$ s are given by

$$\begin{aligned}\tilde{Q}_{0,1} &= \frac{1}{2c} \left(\mu^{-1}K_0\mu + K_0 \log(c^{-1}) + \underline{L}(c^{-2}\mu) \right) |\not{L}\psi|^2, \\ \tilde{Q}_{0,2} &= -c^{-1} \left(X_A(c^{-2}\mu) - \mu^{-1}(\zeta_A + \eta_A) \right) \underline{L}\psi \cdot X_A(\psi), \\ \tilde{Q}_{0,3} &= -c^{-1}\mu^{-1} \left(\zeta_A + \eta_A \right) L\psi X_A(\psi), \\ \tilde{Q}_{0,4} &= -\frac{1}{2} \left(\hat{\chi}_{AB} X^A\psi X^B\psi + \frac{c}{2}\mu^{-1} (\text{tr}_{\tilde{g}}\tilde{\chi} + \text{tr}_{\tilde{g}}\tilde{\chi}) L\psi \cdot \underline{L}\psi \right), \\ \tilde{Q}_{0,5} &= -\frac{1}{2} \frac{1}{c\mu} \underline{L}(c^{-2}\mu) (\underline{L}\psi)^2.\end{aligned}\tag{5.20}$$

For \tilde{Q}_1 we have:

$$\begin{aligned}c^{-2}\tilde{Q}_1 &= -c^{-2}\rho \cdot (K_1\psi - t\psi) - \frac{1}{2}T^{\mu\nu}\tilde{\pi}'_{1,\mu\nu} = \tilde{Q}_{1,0} + \tilde{Q}_{1,1} + \tilde{Q}_{1,2} + \tilde{Q}_{1,3} + \tilde{Q}_{1,4} + \tilde{Q}_{1,5} \\ &= -c^{-2}\rho \cdot (K_1\psi - t\psi) - \frac{1}{2}T^{LL}\tilde{\pi}'_{1,LL} - T^{\underline{L}\underline{L}}\tilde{\pi}'_{1,\underline{L}\underline{L}} - T^{LA}\tilde{\pi}'_{1,LA} - \frac{1}{2}T^{AB}\tilde{\pi}'_{1,AB} - \frac{1}{2}\psi^2 \square_{\tilde{g}}(t).\end{aligned}\tag{5.21}$$

The $\tilde{Q}_{1,i}$ are given by

$$\begin{aligned}
\tilde{Q}_{1,1} &= -\frac{1}{2c\mu} \left(\left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) \underline{L}(c^{-2}\mu) - L \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) \right) (\underline{L}\psi)^2, \\
\tilde{Q}_{1,2} &= \frac{1}{2c} \left(\mu^{-1} K_1 \mu + \underline{L} \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) + 2t - K_1 \log c \right) |\not{d}\psi|^2, \\
\tilde{Q}_{1,3} &= 2\mu^{-1} \left(\frac{2}{c} \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) (\underline{\zeta}_A + \underline{\eta}_A) - \mu \nabla_A \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) \right) (\underline{L}\psi)(X_A\psi), \\
\tilde{Q}_{1,4} &= -\frac{1}{c} \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) \widehat{\chi}_{AB} (X^A\psi)(X^B\psi)
\end{aligned} \tag{5.22}$$

The rest of this section is devoted to the estimates for $\int_{W_{\underline{u}}^t} \tilde{Q}_0$ and $\int_{W_{\underline{u}}^t} \tilde{Q}_1$. We will make use of (5.6).

5.3. Estimates on $\tilde{Q}_{1,2}$. We separate the estimates on $\tilde{Q}_{1,2}$ from others, which are of lower order. We analyze the contribution of each term in the parenthesis in the expression of $\tilde{Q}_{1,2}$. By the definition of K_1 and (4.1) (4.2) (4.4), as well as the fact $\frac{(-t)}{\widetilde{\text{tr}\chi}} \sim t^2$ we have

$$\|K_1 \log c\|_{L^\infty} \lesssim \delta$$

So the contribution of this term to the spacetime integral is bounded by:

$$\delta \int_{-r_0}^t (-t')^{-2} \underline{E}(t', \underline{u}) dt' \tag{5.23}$$

We move to the contribution of the term $\underline{L} \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) + 2t$, a direct calculation implies

$$\underline{L} \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) = -\frac{2}{\widetilde{\text{tr}\chi}} - \frac{2(-t)}{(\widetilde{\text{tr}\chi})^2} \underline{L}(\widetilde{\text{tr}\chi})$$

In view of the expression of $\underline{E}(t, \underline{u})$, the term on the right hand side of above equation behaving like $(-t)$ gives us the borderline contribution. However, we will show these actually cancel. In view of the propagation equation (2.35), the second term on the right hand side of above equation can be written as:

$$\frac{2t}{(\widetilde{\text{tr}\chi})^2} \left(e \text{tr}\chi - \frac{1}{2} (\text{tr}\chi)^2 - |\widehat{\chi}|^2 - \text{tr}\alpha' + O(\delta M^2 (-t)^{-3}) \right)$$

The borderline term is $(-t) \frac{(\text{tr}\chi)^2}{(\widetilde{\text{tr}\chi})^2}$, which can be rewritten as

$$(-t) \frac{(\widetilde{\text{tr}\chi})^2 + O(\delta M^2)(-t)^{-4}}{(\widetilde{\text{tr}\chi})^2},$$

so the borderline term is $(-t)$. On the other hand, in view of (3.24), the borderline term in $-\frac{2}{\widetilde{\text{tr}\chi}}$ is also $(-t)$.

Therefore there is no borderline term in $\underline{L} \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) + 2t$. And the contribution of this term is bounded as

$$\delta M^2 \int_{-r_0}^t (-t')^{-2} \underline{E}(t', \underline{u}) dt'. \tag{5.24}$$

Finally the contribution of the term $\mu^{-1}K_1\mu$ is: In the non-shock region we have, using the fact $\mu \gtrsim \frac{1}{10}$ and $|\underline{L}\mu| \lesssim (-t)^{-2}$,

$$\int_{W_{\underline{u}}^t \cap W_{rare}} \frac{1}{2c} \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) \mu^{-1} \underline{L}\mu \cdot |\not{d}\psi|^2 \lesssim \int_{-r_0}^t (-t')^{-2} \underline{E}(t', \underline{u}) dt'. \quad (5.25)$$

In the shock region, by Proposition 4.6, the spacetime integral

$$\int_{W_{\underline{u}}^t \cap W_{shock}} \frac{1}{2c} \left(\frac{2(-t)}{\widetilde{\text{tr}\chi}} \right) \mu^{-1} \underline{L}\mu \cdot |\not{d}\psi|^2 := -K(t, \underline{u}) \quad (5.26)$$

is negative. Combining (5.23)-(5.26), the spacetime integral involving $\tilde{Q}_{1,2}$ is bounded by:

$$\int_{-r_0}^t (-t')^{-2} \underline{E}(t', \underline{u}) dt' - K(t, \underline{u}). \quad (5.27)$$

5.4. Estimates on $\tilde{Q}_{0,1}$. Another term which also needs to be treated separately is $\tilde{Q}_{0,1}$. Again, we will estimate the contribution of each term in parenthesis. In view of (4.2), (4.4), (4.5) and (4.1), we have:

$$\|K_0 \log c\|_{L^\infty(\Sigma_t^u)} \lesssim (-t)^{-2}, \quad \|\underline{L}(c^{-2}\mu)\|_{L^\infty(\Sigma_t^u)} \lesssim (-t)^{-2}.$$

Therefore the corresponding contributions are bounded by:

$$\int_{-r_0}^t (-t')^{-2} \underline{E}(t', \underline{u}) dt' \quad (5.28)$$

There are two different terms in $\mu^{-1}K_0\mu$, $\mu^{-1}\underline{L}\mu$ and $\mu^{-1}L\mu$. We split the contribution from $\mu^{-1}\underline{L}\mu$ as:

$$\int_{W_{\underline{u}}^t \cap W_{shock}} \frac{1}{2c} \mu^{-1} \underline{L}\mu |\not{d}\psi|^2 + \int_{W_{\underline{u}}^t \cap W_{rare}} \frac{1}{2c} \mu^{-1} \underline{L}\mu |\not{d}\psi|^2.$$

The integral in the non-shock region is bounded through (4.5) by:

$$\int_{-r_0}^t (-t')^{-2} \underline{E}(t', \underline{u}) dt' \quad (5.29)$$

While by Proposition 4.6, the integrand in the shock region is actually negative, so it does not contribute. For the contribution from $\mu^{-1}L\mu$, we only need to consider its positive part, namely, $\mu^{-1}(L\mu)_+$, which is bounded by $\mu^{-1}(T\mu)_+ + c^{-2}(\underline{L}\mu)_+$. The contribution from the second term is bounded similarly as (5.29) through (4.5). In view of Proposition 4.15 the contribution of $\mu^{-1}(T\mu)_+$ is bounded as:

$$\int_{-r_0}^t \|\mu^{-1}(T\mu)_+\|_{L^\infty(\Sigma_t^u)} (-t')^{-2} \underline{E}(t', \underline{u}) dt' \lesssim \delta^{-1} \int_{-r_0}^t \frac{1}{|t' - s|^{1/2}} (-t')^{-3/2} \underline{E}(t', \underline{u}) dt'. \quad (5.30)$$

5.5. Estimates for $\tilde{Q}_{1,1}$. In order to estimate the contribution of $\tilde{Q}_{1,1}$ to the spacetime error integral, we use the trace of the structure equation (2.32) and (2.35):

$$\begin{aligned} T(\text{tr}\chi) + \frac{1}{2}c^{-1}\mu(\theta_C^B \chi_B^C + \theta_C^A \chi_A^C) &= \text{div}\eta + \mu^{-1}\not{g}^{AB} \zeta_A \eta_B - c^{-1}\underline{L}(c^{-1}\mu)\text{tr}\chi, \\ \underline{L}(\text{tr}\chi) + |\chi|_{\not{g}}^2 &= \text{etr}\chi - \text{tr}\alpha'. \end{aligned} \quad (5.31)$$

In view of the formula for $\underline{L}\mu$ and the pointwise estimate for $\widetilde{\text{tr}\chi}$, the first term in the parenthesis of the first formula in (5.22) is bounded by an absolute constant. For the second term, recall that $L = c^{-2}\mu\underline{L} + 2T$. Therefore

$$\begin{aligned}
L\left(\frac{2(-t)}{\widetilde{\text{tr}}\underline{\chi}}\right) &= c^{-2}\mu\underline{L}\left(\frac{2(-t)}{\widetilde{\text{tr}}\underline{\chi}}\right) + 2T\left(\frac{2(-t)}{\widetilde{\text{tr}}\underline{\chi}}\right) \\
&= -\frac{2c^{-2}\mu}{\widetilde{\text{tr}}\underline{\chi}} - \frac{2(-t)c^{-2}\mu\underline{L}(\widetilde{\text{tr}}\underline{\chi})}{(\widetilde{\text{tr}}\underline{\chi})^2} - \frac{4(-t)T(\widetilde{\text{tr}}\underline{\chi})}{(\widetilde{\text{tr}}\underline{\chi})^2} \\
&=: I + II + III.
\end{aligned} \tag{5.32}$$

First, the contribution of I to $\widetilde{Q}_{1,1}$ is negative, so we ignore it. For II and III , their contributions from the right hand side of the two equations in (5.31) are bounded by an absolute constant. We focus on the contributions from the second term in each of (5.31). Since the difference between $\widetilde{\text{tr}}\underline{\chi}$ and $\text{tr}\underline{\chi}$ is lower order, we work the original metric $\not\chi$. In II this contribution is

$$\frac{2(-t)c^{-2}\mu|\underline{\chi}|_{\not\chi}^2}{(\widetilde{\text{tr}}\underline{\chi})^2}. \tag{5.33}$$

In view of the fact $\underline{\chi} = -c\theta$, this contribution from III is

$$-\frac{4(-t)c^{-2}\mu|\underline{\chi}|_{\not\chi}^2}{(\widetilde{\text{tr}}\underline{\chi})^2}. \tag{5.34}$$

(5.33) and (5.34) together give a negative contribution. Therefore this contribution is ignored. Therefore the spacetime error integral contributed by $Q_{1,1}$ is bounded by (up to a constant)

$$\int_{-r_0}^t \int_0^{\underline{u}} \int_{S_{t',\underline{u}'}} \left(\underline{L}\psi + \frac{1}{2}\widetilde{\text{tr}}\underline{\chi}\psi \right)^2 d\mu_{\not\chi} d\underline{u}' dt' + \int_{-r_0}^t \int_0^{\underline{u}} \int_{S_{t',\underline{u}'}} \left(\frac{1}{2}\widetilde{\text{tr}}\underline{\chi}\psi \right)^2 d\mu_{\not\chi} d\underline{u}' dt'. \tag{5.35}$$

The first term above is bounded by

$$\int_0^{\underline{u}} \int_{-r_0}^t (-t')^2 \int_{S_{t',\underline{u}'}} \left(\underline{L}\psi + \frac{1}{2}\widetilde{\text{tr}}\underline{\chi}\psi \right)^2 d\mu_{\not\chi} d\underline{u}' dt' \lesssim \int_0^{\underline{u}} \underline{F}(t, \underline{u}') d\underline{u}'. \tag{5.36}$$

In view of (5.16), the second term in (5.35) is bounded by (up to a constant)

$$\begin{aligned}
&\int_{-r_0}^t \int_0^{\underline{u}} \int_{S_{t',\underline{u}'}} \left(\frac{1}{2}\widetilde{\text{tr}}\underline{\chi}\psi \right)^2 d\mu_{\not\chi} d\underline{u}' dt' \\
&\lesssim \delta^2 \int_{-r_0}^t (-t')^{-2} E(t', \underline{u}) dt' + \delta \int_{-r_0}^t (-t')^{-2} \int_0^{\underline{u}} \int_{S_{t',\underline{u}'}} (\underline{L}\psi)^2 d\mu_{\not\chi} d\underline{u}' dt' \\
&\lesssim \delta^2 \int_{-r_0}^t (-t')^{-2} E(t', \underline{u}) dt' + \delta \int_0^{\underline{u}} \underline{F}(t, \underline{u}') d\underline{u}' + \delta \int_{-r_0}^t (-t')^{-2} \int_0^{\underline{u}} \int_{S_{t',\underline{u}'}} \left(\frac{1}{2}\widetilde{\text{tr}}\underline{\chi}\psi \right)^2 d\mu_{\not\chi} d\underline{u}' dt'.
\end{aligned} \tag{5.37}$$

If δ is appropriately small, we have

$$\begin{aligned}
& \int_{-r_0}^t \int_0^{\underline{u}} \int_{S_{t', \underline{u}'}} \left(\frac{1}{2} \widetilde{\text{tr}} \chi \psi \right)^2 d\mu_g d\underline{u}' dt' \\
& \lesssim \delta^2 \int_{-r_0}^t (-t')^{-2} E(t', \underline{u}) dt' + \delta \int_0^{\underline{u}} \underline{F}(t, \underline{u}') d\underline{u}'.
\end{aligned} \tag{5.38}$$

5.6. Estimates for the other error terms.

5.6.1. *Estimates for $\tilde{Q}_{0,2}$.* The terms in the parenthesis can be written as

$$\mu^{-1} \not{d}\mu + O((-t)^{-2})$$

The contribution from the second term is bounded as:

$$\int_{-r_0}^t (-t')^{-2} \int_0^{\underline{u}} \int_{S_{t', \underline{u}'}} \mu (\underline{L}\psi)^2 d\mu_g d\underline{u}' dt' + \int_{-r_0}^t (-t')^{-2} \int_0^{\underline{u}} \int_{S_{t', \underline{u}'}} \mu |\not{d}\psi|^2 d\mu_g d\underline{u}' dt' := I + II$$

In view of the definition of $\underline{E}(t, \underline{u})$,

$$|II| \lesssim \int_{-r_0}^t (-t')^{-2} \underline{E}(t', \underline{u}) dt'. \tag{5.39}$$

While I is bounded as

$$\begin{aligned}
|I| & \lesssim \int_{-r_0}^t (-t')^{-2} \int_0^{\underline{u}} \int_{S_{t', \underline{u}'}} \mu \left((\underline{L}\psi + \frac{1}{2} \widetilde{\text{tr}} \chi \psi) \right)^2 d\mu_g d\underline{u}' dt' \\
& + \int_{-r_0}^t (-t')^{-2} \int_0^{\underline{u}} \int_{S_{t', \underline{u}'}} \mu \left(\frac{1}{2} \widetilde{\text{tr}} \chi \psi \right)^2 d\mu_g d\underline{u}' dt' \\
& \lesssim \int_{-r_0}^t (-t')^{-2} \underline{E}(t', \underline{u}) dt' + \int_{-r_0}^t (-t')^{-4} E(t', \underline{u}) dt'.
\end{aligned} \tag{5.40}$$

where in the last step we used (5.16). The contribution of $\mu^{-1} \not{d}\mu$ is bounded as:

$$\int_{-r_0}^t (-t')^{-2} \int_0^{\underline{u}} \int_{S_{t', \underline{u}'}} (\underline{L}\psi)^2 d\mu_g d\underline{u}' dt' + \int_{W_{\underline{u}}^t} (-t')^{-2} \mu^{-1} |\not{d}\psi|^2 d\mu_g$$

The first term above is bounded as:

$$\int_0^{\underline{u}} \int_{-r_0}^t (-t')^{-2} \int_{S_{t', \underline{u}'}} (\underline{L}\psi)^2 d\mu_g dt' d\underline{u}' \lesssim \int_0^{\underline{u}} F(t, \underline{u}') d\underline{u}'. \tag{5.41}$$

While the second term is split as:

$$\int_{W_{\underline{u}}^t \cap W_{rare}} (-t')^{-2} \mu^{-1} |\not{d}\psi|^2 d\mu_g + \int_{W_{\underline{u}}^t \cap W_{shock}} (-t')^{-2} \mu^{-1} |\not{d}\psi|^2 d\mu_g,$$

The contribution in non-shock region is bounded as:

$$\int_{-r_0}^t (-t')^{-2} \int_0^{\underline{u}} \int_{S_{t', \underline{u}'}} \mu |\not{d}\psi|^2 d\mu_g d\underline{u}' dt' \lesssim \int_{-r_0}^t (-t')^{-2} \underline{E}(t', \underline{u}) dt'. \tag{5.42}$$

The contribution in shock region is bounded as:

$$\int_{W_{\underline{u}}^t \cap W_{shock}} \mu^{-1} (-t')^{-2} |\not{d}\psi| d\mu_g \lesssim K(t, \underline{u}). \tag{5.43}$$

Here we used the fact that in the shock region $\underline{L}\mu \lesssim -(-t)^{-2}$. This completes the estimates for $\tilde{Q}_{0,2}$.

5.6.2. *Estimates for $\tilde{Q}_{0,3}$.* The estimates for $\tilde{Q}_{0,3}$ is similar to $\tilde{Q}_{0,2}$. Its contribution is bounded by:

$$\int_{-r_0}^t (-t')^{-2} E(t', \underline{u}) dt' + \int_{-r_0}^t (-t')^{-2} \underline{E}(t', \underline{u}) dt' + K(t, \underline{u}). \quad (5.44)$$

5.6.3. *Estimates for $\tilde{Q}_{0,4}$.* In view of (3.24), the contribution of the first term is bounded as:

$$\delta M^2 \int_{-r_0}^t (-t')^{-3} \int_0^{\underline{u}} \int_{S_{t', \underline{u}'}} \mu |\not{d}\psi|^2 d\mu_{\not{g}} d\underline{u}' dt' \lesssim \delta M^2 \int_{-r_0}^t (-t')^{-3} \underline{E}(t', \underline{u}) dt'. \quad (5.45)$$

For the contribution of the second term, in view of (2.28), we have $\tilde{\chi} = c^{-2} \mu \tilde{\chi} + 2c^{-1} \mu \theta = c^{-2} \mu \tilde{\chi} - 2c^{-2} \mu \tilde{\chi} = -c^{-2} \mu \tilde{\chi}$. Therefore in view of (3.24) and (4.6), we have:

$$\|\tilde{\text{tr}} \tilde{\chi} + \tilde{\text{tr}} \tilde{\chi}\|_{L^\infty(\Sigma_t^{\underline{u}})} = \|\tilde{\text{tr}} \tilde{\chi} (1 - c^{-2} \mu)\|_{L^\infty(\Sigma_t^{\underline{u}})} \lesssim (-t)^{-2}$$

So the contribution of the second term in $\tilde{Q}_{0,4}$ is bounded as:

$$\begin{aligned} & \int_{-r_0}^t (-t')^{-2} \int_0^{\underline{u}} \int_{S_{t', \underline{u}'}} (L\psi)^2 d\mu_{\not{g}} d\underline{u}' dt' \\ & + \int_0^{\underline{u}} \int_{-r_0}^t (-t')^{-2} \int_{S_{t', \underline{u}'}} (\underline{L}\psi)^2 d\mu_{\not{g}} dt' d\underline{u}' \\ & \lesssim \int_{-r_0}^t (-t')^{-2} E(t', \underline{u}) dt' + \int_0^{\underline{u}} F(t, \underline{u}') d\underline{u}'. \end{aligned} \quad (5.46)$$

5.6.4. *Estimates for $\tilde{Q}_{0,5}$.* The estimates for $\tilde{Q}_{0,5}$ is straightforward. In view of (4.5), we have

$$\|\underline{L}(c^{-2} \mu)\|_{L^\infty(\Sigma_t^{\underline{u}})} \lesssim (-t)^{-2}.$$

So this contribution is bounded by:

$$\int_0^{\underline{u}} \int_{-r_0}^t (-t')^{-2} \int_{S_{t', \underline{u}'}} (\underline{L}\psi)^2 d\mu_{\not{g}} dt' d\underline{u}' \lesssim \int_0^{\underline{u}} F(t, \underline{u}') d\underline{u}'. \quad (5.47)$$

5.6.5. *Estimates for $\tilde{Q}_{1,3}$.* The first term in the parenthesis together with the factor μ^{-1} can be written as

$$\mu^{-1} \not{d}\mu O((-t)^2) + O(\delta M^2 (-t)^{-1}),$$

The contribution of the second factor above is bounded as:

$$\begin{aligned} & \delta M^2 \int_{-r_0}^t (-t')^{-1} \int_0^{\underline{u}} \int_{S_{t', \underline{u}'}} \mu |\not{d}\psi|^2 d\mu_{\not{g}} d\underline{u}' dt' \\ & + \delta M^2 \int_{-r_0}^t (-t')^{-1} \int_0^{\underline{u}} \int_{S_{t', \underline{u}'}} \mu \left(\underline{L}\psi + \frac{1}{2} \tilde{\text{tr}} \tilde{\chi} \psi \right)^2 d\mu_{\not{g}} d\underline{u}' dt' \\ & + \delta M^2 \int_{-r_0}^t (-t')^{-3} \int_0^{\underline{u}} \int_{S_{t', \underline{u}'}} \psi^2 d\mu_{\not{g}} d\underline{u}' dt'. \end{aligned}$$

The first two terms above can be bounded as

$$\delta M^2 \int_{-r_0}^t (-t')^{-3} \underline{E}(t', \underline{u}) dt'. \quad (5.48)$$

While in view of (5.16), the last term is bounded as

$$\delta^3 M^2 \int_{-r_0}^t (-t')^{-3} E(t', \underline{u}) dt' \quad (5.49)$$

For the contribution of the factor $\mu^{-1}\not{d}\mu O((-t)^2)$, we bound the term $(\underline{L}\psi)(X_A\psi)$ as:

$$|(\underline{L}\psi)(X_A\psi)| \leq \epsilon |\not{d}\psi|^2 + C_\epsilon (\underline{L}\psi)^2,$$

where ϵ is a small absolute positive constant which will be determined later. The contribution of the first term on the right hand side above is bounded as:

$$\begin{aligned} & \epsilon \int_{W_{\underline{u}}^t \cap W_{rare}} |\not{d}\psi|^2 + \epsilon \int_{W_{\underline{u}}^t \cap W_{shock}} \mu^{-1} |\not{d}\psi|^2 \\ & \leq \epsilon \int_{-r_0}^t (-t')^{-2} \underline{E}(t', \underline{u}) dt' + \epsilon K(t, \underline{u}). \end{aligned} \quad (5.50)$$

The contribution from $(\underline{L}\psi)^2$ can be bounded as:

$$\begin{aligned} & C_\epsilon \int_{-r_0}^t \int_0^{\underline{u}} \int_{S_{t', \underline{u}'}} \left(\underline{L}\psi + \frac{1}{2} \widetilde{\text{tr}} \chi \psi \right)^2 d\mu_{\not{g}} du' dt' + \int_{-r_0}^t (-t')^{-2} \int_0^{\underline{u}} \int_{S_{t', \underline{u}'}} \psi^2 d\mu_{\not{g}} du' dt' \\ & \leq C_\epsilon \int_{-r_0}^t (-t')^{-2} \underline{E}(t', \underline{u}) dt' + C_\epsilon \delta^2 \int_{-r_0}^t (-t')^{-2} E(t', \underline{u}) dt'. \end{aligned} \quad (5.51)$$

In view of (4.9), the second term in the parenthesis of $\tilde{Q}_{1,3}$ together with the factor μ^{-1} is bounded by $(-t)^{-1}$. Therefore the contribution of this term is bounded, in the same process as we derive (5.51), by:

$$\int_{-r_0}^t (-t')^{-3} \underline{E}(t', \underline{u}) dt' + \delta^2 \int_{-r_0}^t (-t')^{-3} E(t', \underline{u}) dt'. \quad (5.52)$$

5.6.6. *Estimates for $\tilde{Q}_{1,4}$.* In view of (3.24), this term is bounded by:

$$\delta M^2 \int_{-r_0}^t (-t')^{-3} \underline{E}(t', \underline{u}) dt'. \quad (5.53)$$

5.6.7. *Estimates for $\tilde{Q}_{1,5}$.* Finally we consider the contribution from $\tilde{Q}_{1,5}$. Writing the operator $\square_{\tilde{g}}$ in the null frame (see (4.1)), we have:

$$\square_{\tilde{g}}(t) = -\frac{1}{2} \text{tr} \tilde{\chi} L t - \frac{1}{2} \text{tr} \tilde{\chi} \underline{L}(t) = -\frac{1}{2} \text{tr} \tilde{\chi} (c^{-2} \mu) - \frac{1}{2} \text{tr} \tilde{\chi} = -(c^{-2} \mu - c^{-1}) \text{tr} \tilde{\chi}$$

In view of (4.6),

$$|(c^{-2} \mu - c^{-1}) \text{tr} \tilde{\chi}| \lesssim (-t)^{-2}$$

Therefore the contribution of $\tilde{Q}_{1,5}$ is bounded by:

$$\delta^2 \int_{-r_0}^t (-t')^{-2} E(t', \underline{u}) dt'. \quad (5.54)$$

5.7. **Conclusion.** In view of (5.5) and the estimates for spacetime integral of \tilde{Q}_0 (5.28)-(5.30), (5.39)-(5.43), (5.44), (5.45)-(5.46), (5.47), we have:

$$\begin{aligned} E(t, \underline{u}) + F(t, \underline{u}) & \leq E(-r_0, \underline{u}) + C \int_0^{\underline{u}} F(t, \underline{u}') d\underline{u}' + C \int_{-r_0}^t (-t')^{-2} \underline{E}(t', \underline{u}) dt' + \int_{W_{\underline{u}}^t} |\rho \cdot K_0 \psi| \\ & \quad + CK(t, \underline{u}) + C \int_{-r_0}^t (-t')^{-2} E(t', \underline{u}) dt' + C \delta^{-1} \int_{-r_0}^t \frac{1}{|t' - s|^{1/2}} (-t')^{-3/2} \underline{E}(t', \underline{u}) dt'. \end{aligned} \quad (5.55)$$

Here $s \in [-2, -1]$ and $-r_0 \leq t' \leq t < s$. The integral in the last term on the right hand side above can be bounded as follows:

$$\begin{aligned}
& \int_{-r_0}^t \frac{(-t')^{-3/2}}{|t' - s|^{1/2}} \underline{E}(t', \underline{u}) dt' = \int_{-r_0}^{2s} \frac{(-t')^{-3/2}}{|t' - s|^{1/2}} \underline{E}(t', \underline{u}) dt' + \int_{2s}^t \frac{(-t')^{-3/2}}{|t' - s|^{1/2}} \underline{E}(t', \underline{u}) dt' \\
& \lesssim \int_{-r_0}^t (-t')^{-3/2} \underline{E}(t', \underline{u}) dt' + \int_{2s}^t \frac{1}{|t' - s|^{1/2}} \underline{E}(t', \underline{u}) dt'.
\end{aligned} \tag{5.56}$$

Using Gronwall we have

$$\begin{aligned}
E(t, \underline{u}) + F(t, \underline{u}) & \leq E(-r_0, \underline{u}) + C\delta^{-1} \int_{-r_0}^t (-t')^{-3/2} \underline{E}(t', \underline{u}) dt' + \left| \int_{W_{\underline{u}}^t} \rho \cdot K_0 \psi \right| \\
& + CK(t, \underline{u}) + C\delta^{-1} \int_{2s}^t \frac{1}{|t' - s|^{1/2}} \underline{E}(t', \underline{u}) dt'.
\end{aligned} \tag{5.57}$$

On the other hand, in view of (5.15) and the estimates for spacetime integral of \tilde{Q}_1 (5.27), (5.36)-(5.38), (5.48)-(5.52), (5.53), (5.54) as well as (5.16), we have:

$$\begin{aligned}
\underline{E}(t, \underline{u}) + \underline{F}(t, \underline{u}) + K(t, \underline{u}) & \leq \underline{E}(-r_0, \underline{u}) + \left| \int_{W_{\underline{u}}^t} \rho \cdot (K_1 \psi - t\psi) \right| + C \int_0^{\underline{u}} \underline{F}(t, \underline{u}') d\underline{u}' \\
& + \epsilon K(t, \underline{u}) + C \int_{-r_0}^t (-t')^{-3/2} \underline{E}(t', \underline{u}) dt' + C\delta^2 \int_{-r_0}^t (-t')^{-2} E(t', \underline{u}) dt'.
\end{aligned} \tag{5.58}$$

Using Gronwall we obtain

$$\underline{E}(t, \underline{u}) + \underline{F}(t, \underline{u}) + K(t, \underline{u}) \leq \underline{E}(-r_0, \underline{u}) + \left| \int_{W_{\underline{u}}^t} \rho \cdot (K_1 \psi - t\psi) \right| + C\delta^2 \int_{-r_0}^t (-t')^{-2} E(t', \underline{u}) dt'. \tag{5.59}$$

Substituting (5.59) into (5.57) we obtain

$$\begin{aligned}
E(t, \underline{u}) + F(t, \underline{u}) & \lesssim E(-r_0, \underline{u}) + \delta^{-1} \underline{E}(-r_0, \underline{u}) + \delta \int_{-r_0}^t (-t')^{-2} E(t', \underline{u}) dt' \\
& + \sup_{t' \in [-r_0, t]} \left| \int_{W_{\underline{u}}^{t'}} \rho \cdot (K_1 \psi - t\psi) \right| + \left| \int_{W_{\underline{u}}^t} \rho \cdot K_0 \psi \right|,
\end{aligned} \tag{5.60}$$

which implies

$$\begin{aligned}
E(t, \underline{u}) + F(t, \underline{u}) & \lesssim E(-r_0, \underline{u}) + \delta^{-1} \underline{E}(-r_0, \underline{u}) \\
& + \sup_{t' \in [-r_0, t]} \left| \int_{W_{\underline{u}}^{t'}} \rho \cdot (K_1 \psi - t\psi) \right| + \left| \int_{W_{\underline{u}}^t} \rho \cdot K_0 \psi \right|.
\end{aligned} \tag{5.61}$$

Substituting (5.61) in (5.59) we obtain

$$\begin{aligned}
\underline{E}(t, \underline{u}) + \underline{F}(t, \underline{u}) + K(t, \underline{u}) & \lesssim \underline{E}(-r_0, \underline{u}) + \delta^2 E(-r_0, \underline{u}) \\
& + \sup_{t' \in [-r_0, t]} \left| \int_{W_{\underline{u}}^{t'}} \rho \cdot (K_1 \psi - t\psi) \right| + \delta^2 \sup_{t' \in [-r_0, t]} \left| \int_{W_{\underline{u}}^{t'}} \rho \cdot K_0 \psi \right|.
\end{aligned} \tag{5.62}$$

Remark 5.5. Here we remark that the spacetime integral involving K_1 in (5.61) and the spacetime integral involving K_0 in (5.62) are lower order compared to the other spacetime integral in the corresponding inequality. More specifically, in (5.61), if we disregard the behavior with respect to t , $K_1\psi - t\psi \sim \delta^{1/2}$, $K_0\psi \sim \delta^{-1/2}$, so the term involving K_0 is lower order. Similar argument applies to (5.62) in which the factor δ^2 makes the term involving K_0 lower order.

6. ESTIMATES FOR RECTANGULAR COORDINATES AS FUNCTIONS OF OPTICAL COORDINATES, ESTIMATES FOR NON-TOP ORDER TERMS

Our energy estimates for (\star) is with respect to the optical coordinates $(t, \underline{u}, \theta)$. In order to go back to the rectangular coordinates (t, x^i) , one needs to investigate the relation between the rectangular coordinates (t, x^i) and the optical coordinates $(t, \underline{u}, \theta)$. In what follows, we will consider x^i s as functions of $(t, \underline{u}, \theta)$ and estimate their derivatives with respect to optical coordinates. In the meantime, we also estimate the quantities y^k, z^k, λ_i and their derivatives in optical coordinates. As a by product, we will also obtain estimates for the lower order objects, i.e., with order $< N_{\text{top}} + 1$.

Given a vectorfield V , we define the null components of its deformation tensor as

$${}^{(V)}\underline{Z}_A = {}^{(V)}\pi(\underline{L}, X_A), \quad {}^{(V)}Z_A = {}^{(V)}\pi(L, X_A), \quad {}^{(V)}\not\!X_{AB} = {}^{(V)}\pi(X_A, X_B). \quad (6.1)$$

The projection of Lie derivative \mathcal{L}_V to $S_{t, \underline{u}}$ is denoted as \mathcal{L}_V . The shorthand notation $\mathcal{L}_{Z_i}^\alpha$ to denote $\mathcal{L}_{Z_{i_1}} \mathcal{L}_{Z_{i_2}} \cdots \mathcal{L}_{Z_{i_k}}$ for a multi-index $\alpha = (i_1, \dots, i_k)$. We will show that, for all $|\alpha| \leq N_\infty$, we have $(-t)\mathcal{L}_{Z_i}^\alpha \underline{X}', \mathcal{L}_{Z_i}^\alpha (Z_j) \underline{Z}, \mathcal{L}_{Z_i}^\alpha (Z_j) \not\!X, (-t)^{-1} \mathcal{L}_{Z_i}^\alpha (Q) (\not\!X + 4), Z_i^{\alpha+1} y^j, (-t)^{-1} Z_i^{\alpha+1} \lambda_j \} \subset \mathcal{O}_{2-2l, 2}^{|\alpha|+1}$, where l is the number of T 's in Z_i 's, $|\alpha| \geq 1$ and $Z_j \neq T$. If $Z_j = T$, then we have $(-t)\mathcal{L}_{Z_i}^\alpha (T) \underline{Z}, \mathcal{L}_{Z_i}^\alpha (T) \not\!X \in \mathcal{O}_{-2l, 1}^{|\alpha|+1}$. Similarly, we will derive L^2 -estimates on objects of order $\leq N_\mu$. The L^2 estimates depend on the L^∞ estimates up to order $N_\infty + 2$. In the course of the proof, it will be clear why N_∞ is chosen to be approximately $\frac{1}{2}N_{\text{top}}$.

6.1. L^∞ estimates. We assume (B.1): for all $|\alpha| \leq N_\infty$, $Z_i^{\alpha+2}\psi \in \Psi_{1-2l, 1}^{|\alpha|+2}$.

Proposition 6.1. For sufficiently small δ , for all $|\alpha| \leq N_\infty$ and $t \in [-r_0, s^*]$, we can bound $\{(-t)\mathcal{L}_{Z_i}^\alpha \underline{X}', \mathcal{L}_{Z_i}^\alpha (Z_j) \underline{Z}, \mathcal{L}_{Z_i}^\alpha (Z_j) \not\!X, (-t)^{-1} \mathcal{L}_{Z_i}^\alpha (Q) (\not\!X + 4), Z_i^{\alpha+1} y^j, (-t)^{-1} Z_i^{\alpha+1} \lambda_j \} \subset \mathcal{O}_{2-2l, 2}^{|\alpha|+1}$ and $\{(-t)^{-2} Z_i^{\alpha+2} x^j, (-t)^{-1} Z_i^{\alpha+1} \widehat{T}^j, (-t)^{-2} \mathcal{L}_{Z_i}^{\alpha+1} Z_j, \mathcal{L}_{Z_i}^\alpha (T) \underline{Z}, \mathcal{L}_{Z_i}^\alpha (T) \not\!X \} \subset \mathcal{O}_{-2l, 1}^{|\alpha|+1}$ in terms of $Z_i^{\alpha+1}\psi$ which belongs to $\Psi_{1-2l, 1}^{|\alpha|+2}$. Here l is the number of T 's in Z_i 's.

Proof. We do induction on the order. When $|\alpha| = 0$, the estimates are treated in Section 3. Here we only treat the estimates when $l = 0$, when $l \geq 1$, we can use the structure equation (2.32) to reduce the problem to the estimates for μ , which is treated in Proposition 6.2. Given $|\alpha| \leq N_\infty$, we assume that estimates hold for terms of order $\leq |\alpha|$. In particular, we have $\mathcal{L}_{R_i}^{\beta-1} \underline{X}', R_i^\beta y^j \in \mathcal{O}_{2, 2}^{|\beta|+1}$ for all $|\beta| \leq |\alpha|$. We prove the proposition for $|\alpha| + 1$.

Step 1 Bounds $R_i^{\alpha+1} x^j$. Let $\delta_{\alpha+1, i}^j = \Omega_i^{\alpha+1} x^j - R_i^{\alpha+1} x^j$ where Ω_i 's are the standard rotational vectorfields on Euclidean space. It is obvious that $\Omega_i^\alpha x^j$ is equal to some x^i , therefore, bounded by r . Since $R_i = \Omega_i - \lambda_i \widehat{T}^j \partial_j$, $R_i x^j \in \mathcal{O}_{0, -1}^0$ and by ignoring all the numerical constants, we have

$$\delta_{\alpha+1, i}^j = R_i^\alpha \left(\lambda_i \widehat{T}^j \right) = R_i^\alpha \left(\lambda_i \left(\frac{x^j}{\underline{u} - t} + y^j \right) \right).$$

Here the index i is not a single index. It means we apply a string of different R_i s. This notation applies in the following when a string of R_i s are considered.

By the induction hypothesis, the right hand side above is in $\mathcal{O}_{2, 1}^{|\alpha|}$. Therefore $R_i^{\alpha+1} x^j \in \mathcal{O}_{0, -1}^{|\alpha|}$.

Step 2 Bounds on $\mathcal{L}_{R_i}^\alpha \underline{\chi}'$. We commute $\mathcal{L}_{R_i}^\alpha$ with (3.20) to derive

$$\begin{aligned} \mathcal{L}_{\underline{L}} \mathcal{L}_{R_i}^\alpha \underline{\chi}' &= [\mathcal{L}_{\underline{L}}, \mathcal{L}_{R_i}^\alpha] \underline{\chi}' + e \cdot \mathcal{L}_{R_i}^\alpha \underline{\chi}' + \underline{\chi}' \cdot \mathcal{L}_{R_i}^\alpha \underline{\chi}' + \sum_{\substack{|\beta_1|+|\beta_2|=|\alpha| \\ |\beta_2|<|\alpha|}} R_i^{\beta_1} e \cdot \mathcal{L}_{R_i}^{\beta_2} \underline{\chi}' \\ &+ \sum_{\substack{|\beta_1|+|\beta_2|+|\beta_3|=|\alpha| \\ |\beta_1|<|\alpha|, |\beta_3|<|\alpha|}} \mathcal{L}_{R_i}^{\beta_1} \underline{\chi}' \cdot \mathcal{L}_{R_i}^{\beta_2} \mathcal{G}^{-1} \cdot \mathcal{L}_{R_i}^{\beta_3} \underline{\chi}' + \mathcal{L}_{R_i}^\alpha \left(\frac{e \mathcal{G}_{AB}}{t - \underline{u}} - \underline{\alpha}'_{AB} \right). \end{aligned} \quad (6.2)$$

Since $e = c^{-1} \frac{d\mathcal{C}}{d\rho} \underline{L}\rho$, $R_i^\alpha e$ is of order $\leq |\alpha| + 1$. By (B.1), we have $R_i^\beta e \in \Psi_{2,3}^{|\alpha|+1}$. Similarly, by the explicit formula of $\underline{\alpha}'_{AB}$, we have $\mathcal{L}_{R_i}^\beta \underline{\alpha}'_{AB} \in \Psi_{2,4}^{|\alpha|+2}$. Since $^{(R_i)}\pi_{AB} = 2c^{-1} \lambda_i \underline{\chi}_{AB}$ and $\mathcal{L}_{R_i}^{\beta_2} \mathcal{G} = \mathcal{L}_{R_i}^{\beta_2-1} {}^{(R_i)}\not\! \pi_{AB}$, by the estimates derived in previous sections and by the induction hypothesis, we can rewrite (6.2) as

$$\mathcal{L}_{\underline{L}} \mathcal{L}_{R_i}^\alpha \underline{\chi}' = [\mathcal{L}_{\underline{L}}, \mathcal{L}_{R_i}^\alpha] \underline{\chi}' + \mathcal{O}_{2,3}^1 \cdot \mathcal{L}_{R_i}^\alpha \underline{\chi}' + \Psi_{2,4}^{\leq |\alpha|+2}.$$

The commutator can be computed as $[\mathcal{L}_{\underline{L}}, \mathcal{L}_{R_i}^\alpha] \underline{\chi}' = \sum_{|\beta_1|+|\beta_2|=|\alpha|-1} \mathcal{L}_{R_i}^{\beta_1} \mathcal{L}_{R_i}^{\beta_2} \mathcal{L}_{R_i}^\alpha \underline{\chi}'$. Since $^{(R_i)}\underline{Z}_A = -\underline{\chi}'_{AB} R_i^B + \varepsilon_{ijk} z^j X_A^k + \lambda_i \not\! \pi_A(c)$ and $z^j = -\frac{(c-1)x^j}{\underline{u}-t} - cy^j$, the commutator term is of type $\mathcal{O}_{2,2}^1 \cdot \mathcal{L}_{R_i}^{\alpha-1} \underline{\chi}'$. Therefore, we have

$$\mathcal{L}_{\underline{L}} \mathcal{L}_{R_i}^\alpha \underline{\chi}' = \mathcal{O}_{2,3}^1 \cdot \mathcal{L}_{R_i}^\alpha \underline{\chi}' + \Psi_{2,4}^{\leq |\alpha|+2}.$$

By integrating this equation from $-r_0$ to t , the Gronwall's inequality yields $\|\mathcal{L}_{R_i}^\alpha \underline{\chi}'\|_{L^\infty(\Sigma_t^u)} \lesssim_M \delta(-t)^{-3}$.

Step 3 Bounds on $R_i^{\alpha+1} y^j$ and $R_i^{\alpha+1} \lambda_j$. Since $R_i y^j = (-c^{-1} \underline{\chi}_B^A - \frac{\delta_B^A}{\underline{u}-t}) R_i^A \not\! \pi_B x^j$, schematically we have

$$R_i^{\alpha+1} y^j = R_i^\alpha R_k y^j = R_i^\alpha \left(\left(c^{-1} \underline{\chi} + \frac{\delta}{\underline{u}-t} \right) \cdot R_k \cdot \not\! \pi x^j \right).$$

We distribute R_i^α inside the parenthesis by Leibniz rule. (Here again, the index i is not a single index, so we use index k to distinct the last rotation vectorfield.) Therefore, a typical term would be either $\mathcal{L}_{R_i}^{\beta_1} \underline{\chi} \cdot \mathcal{L}_{R_i}^{\beta_2} \mathcal{G}^{-1} \cdot \mathcal{L}_{R_i}^{\beta_3} R_k \cdot \not\! \pi R_i^{\beta_4} x^j$ or $\mathcal{L}_{R_i}^{\beta_1} \mathcal{G} \cdot \mathcal{L}_{R_i}^{\beta_2} \mathcal{G}^{-1} \cdot \mathcal{L}_{R_i}^{\beta_3} R_k \cdot \not\! \pi R_i^{\beta_4} x^j$ with $|\beta_1| + |\beta_2| + |\beta_3| + |\beta_4| = |\alpha|$. There are only two terms where are not included in the induction hypothesis: $\mathcal{L}_{R_i}^{\beta_2} \mathcal{G}_{AB}$ and $\mathcal{L}_{R_i}^{\beta_3} R_j$. The first term is in fact easy to handle by induction hypothesis and estimates derived in Step 1 and Step 2, since $\mathcal{L}_{R_i} \mathcal{G}_{AB} = {}^{(R_i)}\not\! \pi_{AB} = 2\lambda_i c^{-1} \underline{\chi}_{AB}$. For the second one, we use the following expression:

$$\begin{aligned} \mathcal{L}_{R_i} R_j &= - \sum_{k=1}^3 \varepsilon_{ijk} R_k + \lambda_i \left(\frac{c^{-1}-1}{\underline{u}-t} \mathcal{G} - c^{-1} \left(\underline{\chi} + \frac{\mathcal{G}}{\underline{u}-t} \right) \right) R_j - \lambda_j \left(\frac{c^{-1}-1}{\underline{u}-t} \mathcal{G} - c^{-1} \left(\underline{\chi} + \frac{\mathcal{G}}{\underline{u}-t} \right) \right) R_i \\ &\quad - \lambda_i \varepsilon_{jkl} y^k \not\! \pi x^l \cdot \mathcal{G}^{-1} + \lambda_j \varepsilon_{ikl} y^k \not\! \pi x^l \cdot \mathcal{G}^{-1}. \end{aligned}$$

Therefore, $\mathcal{L}_{R_i}^{\beta_3} R_k = \mathcal{L}_{R_i}^{\beta_3-1} \mathcal{L}_{R_i} R_k = \mathcal{O}_{0,-1}^{|\beta_3|-1} + \mathcal{O}_{\geq 2, \geq 2}^{|\beta_3|-1} R_i^{\beta_3} x^j$. Finally, we obtain that

$$R_i^{\alpha+1} y^j = \mathcal{O}_{2,2}^{\leq |\alpha|+1} + \mathcal{O}_{2, \geq 2}^{\leq |\alpha|} \cdot \not\! \pi R_i^{\beta_4} x^j.$$

Although $\not\! \pi R_i^{\beta_4} x^j$ and $\mathcal{L}_{R_i}^{\beta_1} \underline{\chi}$ may have order $|\alpha| + 1$, they have been controlled from previous steps. This gives the bounds on $R_i^{\alpha+1} y^j$. Then by the fact that $\lambda_j = \varepsilon_{jkl} x^k y^l$, the estimate for $R_i^{\alpha+1} \lambda_j$ follows. This completes the proof of the proposition. \square

Proposition 6.2. *For sufficiently small δ , for all $|\alpha| \leq N_\infty, t \in [-r_0, s^*]$, we can bound $Z_i^{\alpha+1} \mu \in \mathcal{O}_{-2l,1}^{|\alpha|+1}$ in terms of $Z_i^{\alpha+2} \psi \in \Psi_{1-2l,1}^{|\alpha|+2}$.*

Proof. We do induction on order. The base case $|\alpha| = 0$ has been treated in Section 3 and Section 4. We assume the proposition holds with order of derivatives on μ at most $|\alpha|$. For $|\alpha| + 1$, by commuting $Z_i^{\alpha+1}$ with $\underline{L}\mu = m + \mu e$, we have

$$\underline{L}\delta^l Z_i^{\alpha+1} \mu = (e + {}^{(Z_i)}\underline{Z})\delta^l Z_i^{\alpha+1} \mu + \delta^l Z_i^{\alpha+1} m + \sum_{\substack{|\beta_1|+|\beta_2|\leq|\alpha|+1 \\ |\beta_1|<|\alpha|}} \delta^{l_1} Z_i^{\beta_1} \mu \delta^{l_2} Z_i^{\beta_2} e$$

where $l_a, a = 1, 2$ is the number of T 's in Z^{β_a} 's and l is the number of T 's in Z^α 's. By the induction hypothesis, the above equation can be written as:

$$\underline{L}\delta^l Z_i^{\alpha+1} \mu = \mathcal{O}_{2,2}^{\leq 1} \cdot \delta^l Z_i^{\alpha+1} \mu + \Psi_{0,2}^{\leq |\alpha|+2}$$

Similar to the estimates derived in the Step 3 in previous section, we can use induction hypothesis and Gronwall's inequality to conclude that $\|Z_i^{\alpha+1} \mu\|_{L^\infty(\Sigma_t)} \lesssim_M \delta^{-l} (-t)^{-1}$. \square

6.2. L^2 estimates. N_{top} will be the total number of derivatives commuted with $\square_{\tilde{g}}\psi = 0$. The highest order objects will be of order $N_{\text{top}} + 1$. In this subsection, based on (B.1) and (B.2), we will derive L^2 estimates on the objects of order $\leq N_{\text{top}}$ in terms of the L^2 norms of $Z_i^{\alpha+2}\psi \in \Psi_{1-2l}^{|\alpha|+2}$ with $|\alpha| \leq N_{\text{top}} - 1$.

Proposition 6.3. *For sufficiently small δ , for all α with $|\alpha| \leq N_{\text{top}} - 1$ and $t \in [-r_0, s^*]$, the $L^2(\Sigma_t^u)$ norms of all the quantities listed below*

$$\mathcal{L}_{Z_i}^\alpha \underline{\chi}', (-t)^{-1} \mathcal{L}_{Z_i}^\alpha {}^{(Z_j)}\underline{Z}, (-t)^{-1} \mathcal{L}_{Z_i}^\alpha {}^{(Z_j)}\not\!Z, (-t)^{-1} Z_i^{\alpha+1} y^j, (-t)^{-2} Z_i^{\alpha+1} \lambda_j,$$

are bounded ³ by $\delta^{1/2-l} \int_{-r_0}^t (-t')^{-3} \mu_m^{-1/2}(t') \sqrt{E_{\leq |\alpha|+2}(t', \underline{u})} dt'$, where l is the number of T 's in Z_i 's.

Proof. We use an induction argument on the order of derivatives. When $|\alpha| = 0$, the result follows from the estimates in Section 3 and 4. Again, here we only treat the case $l = 0$. The case $l \geq 1$ can be treated using (2.32). By assuming the proposition holds for terms with order $\leq |\alpha|$, we show it holds for $|\alpha| + 1$.

Step 1 Bounds on $\mathcal{L}_{R_i}^\alpha \underline{\chi}'$. By affording a \underline{L} -derivative, we have

$$\|\mathcal{L}_{R_i}^\alpha \underline{\chi}'\|_{L^2(\Sigma_t^u)} \lesssim \|\mathcal{L}_{R_i}^\alpha \underline{\chi}'\|_{L^2(\Sigma_{-r_0}^u)} + \int_{-r_0}^t \|\underline{\chi}'\| \|\mathcal{L}_{R_i}^\alpha \underline{\chi}'\| + \|\mathcal{L}_{\underline{L}} \mathcal{L}_{R_i}^\alpha \underline{\chi}'\|_{L^2(\Sigma_\tau^u)} d\tau. \quad (6.3)$$

We use formula (6.2) to replace $\mathcal{L}_{\underline{L}} \mathcal{L}_{R_i}^\alpha \underline{\chi}'$ by the terms with lower orders. Each nonlinear term has at most one factor with order $> N_\infty$. We bound this factor in $L^2(\Sigma_t)$ and the rest in L^∞ . We now indicate briefly how the estimates on the factors involving e and α' work.

For α' , since $\alpha'_{AB} = c \frac{dc}{d\rho} \mathcal{D}_{A,B}^2 \rho + \frac{1}{2} \left[\frac{d^2(c^2)}{d\rho^2} - \frac{1}{2c^2} \left(\frac{dc^2}{d\rho} \right)^2 \right] X_A(\rho) X_B(\rho)$, in view of the definition of $\underline{E}(t, \underline{u})$, for sufficiently small δ , we have

$$\|\mathcal{L}_{R_i}^\alpha \alpha'_{AB}\|_{L^2(\Sigma_t^u)} \lesssim \sum_{|\alpha| \leq k} (-t)^{-2} \|\mathcal{L}_{R_i}^{\alpha+1} \psi\|_{L^2(\Sigma_t^u)} \lesssim_M \delta^{\frac{1}{2}} (-t)^{-3} \mu_m^{-1/2}(t) \sqrt{E_{\leq |\alpha|+2}(t, \underline{u})}.$$

For e , since $e = c^{-1} \frac{dc}{d\rho} \underline{L}\rho$, we have

$$\begin{aligned} \|R_i^\alpha e\|_{L^2(\Sigma_t^u)} &\lesssim_M (-t)^{-2} \sum_{|\beta| \leq |\alpha|} \left(\delta^{1/2} \|\mathcal{L}_{R_i}^{|\beta|-1} \psi\|_{L^2(\Sigma_t^u)} + \delta^{1/2} \|\mathcal{L}_{R_i}^{|\beta|-1} Q\psi\|_{L^2(\Sigma_t^u)} \right) \\ &\lesssim_M \delta^{1/2} (-t)^{-3} \mu_m^{-1/2}(t) \sqrt{E_{\leq |\alpha|+1}(t, \underline{u})}. \end{aligned}$$

³The inequality is up to a constant depending only on the bootstrap constant M .

By applying Gronwall's inequality to (6.3), we obtain immediately that

$$\|\mathcal{L}_{R_i}^\alpha \underline{\chi}'\|_{L^2(\Sigma_t^\pm)} \leq \|\mathcal{L}_{R_i}^\alpha \underline{\chi}'\|_{L^2(\Sigma_{-r_0}^\pm)} + C_M \delta^{1/2} \int_{-r_0}^t (-\tau)^{-3} \mu_m^{-1/2}(\tau) \sqrt{E_{\leq |\alpha|+2}(\tau, \underline{u})} d\tau$$

Step 2 Bounds on $R_i^{\alpha+1} y^j$. By the computations in the Step 2 of the proof of Proposition 6.1, $R_i^{\alpha+1} y$ is a linear combination of the terms such as $\mathcal{L}_{R_i}^{\beta_1} \underline{\chi} \cdot \mathcal{L}_{R_i}^{\beta_2} \mathcal{G}^{-1} \cdot \mathcal{L}_{R_i}^{\beta_3} R_j \cdot \mathcal{L}_{R_i}^{\beta_4} x^j$ or as $\mathcal{L}_{R_i}^{\beta_1} \mathcal{G} \cdot \mathcal{L}_{R_i}^{\beta_2} \mathcal{G}^{-1} \cdot \mathcal{L}_{R_i}^{\beta_3} R_j \cdot \mathcal{L}_{R_i}^{\beta_4} x^j$, where $|\beta_1| + |\beta_2| + |\beta_3| + |\beta_4| = |\alpha|$. Similarly, we bound all factors with order $\leq N_\infty$ by the L^∞ estimates in Proposition 6.1. By the induction hypothesis, this yields the bound on $R_i^{\alpha+1} y^j$ immediately. The estimates for other quantities follow from the estimates of $\underline{\chi}'$ and y^j . Again, as in [18], In this process, the terms like $R_i^{\beta} x^j$ and the leading term in $\mathcal{L}_{R_i} R_j$, which can be bounded by a constant $C(-t)$ disregarding the order of the derivatives, are bounded in L^∞ . The rest terms in $\mathcal{L}_{R_i} R_j$, which depend on $\underline{\chi}'$ and y^j as well as their derivatives, are bounded in L^2 based on the L^2 estimates for $\mathcal{L}_{R_i}^\beta \underline{\chi}'$. \square

We also have L^2 estimates for derivatives of μ .

Proposition 6.4. *For sufficiently small δ , for all α with $|\alpha| \leq N_{\text{top}} - 1$ and $t \in [-r_0, s^*]$, we have*

$$\begin{aligned} \delta^l (-t)^{-1} \|Z_i^{\alpha+1} \mu\|_{L^2(\Sigma_t^\pm)} &\lesssim_M \delta^l (-r_0)^{-1} \|Z_i^{\alpha+1} \mu\|_{L^2(\Sigma_{-r_0}^\pm)} \\ &\quad + \delta^{1/2} \int_{-r_0}^t (-\tau)^{-2} \left(\sqrt{E_{\leq |\alpha|+2}(\tau, \underline{u})} + \mu_m^{-1/2}(\tau) (-\tau)^{-1} \sqrt{E_{\leq |\alpha|+2}(\tau, \underline{u})} \right) d\tau. \end{aligned}$$

Proof. According to the proof of Proposition 6.2, we have

$$\begin{aligned} \delta^l \underline{L} \left((-t)^{-1} |Z_i^{\alpha+1} \mu| \right) &\lesssim \delta^l (-t)^{-1} |Z_i^{\alpha+1} m| + \delta^l (-t)^{-2} |Z_i^{\alpha+1} \mu| \\ &\quad + \delta^l (|e| + |(R_i) \underline{Z}|) (-t)^{-1} |Z_i^{\alpha+1} \mu| \\ &\quad + \sum_{|\beta_1|+|\beta_2| \leq |\alpha|} \delta^{l_1} (-t)^{-1} |Z_i^{\beta_1} \mu| \delta^{l_2} |R_i^{\beta_2} e|. \end{aligned}$$

Then the result follows by using Gronwall and an induction argument, which are similar as in the proof of Proposition 6.2. \square

7. ESTIMATES FOR THE TOP ORDER OPTICAL TERMS

As we already stated, the highest possible order of an object in the paper will be $N_{\text{top}} + 1$. The current section is devoted to the L^2 estimates of $\mathcal{L} X_A R_i^{\alpha-1} \text{tr} \underline{\chi}$ and $X_A Z_i^{\alpha+1} \mu$ with $|\alpha| = N_{\text{top}} - 1$. Here we choose X_A as the first member of the string of commutators because this avoids a logarithmic divergence in the estimates. When there is no confusion, we use the notation $\mathcal{R}_i^\alpha := X_A R_i^{\alpha-1}$.

7.1. Estimates for the contribution from $\text{tr} \underline{\chi}$. Since we deal with top order terms, we can not use the transport equation (2.33) directly as in the previous section, which causes a loss of derivative. Instead, we derive an elliptic system coupled with a transport equation for $\widehat{\chi}$ and $\mathcal{L} \text{tr} \underline{\chi}$:

$$\underline{L}(\mathcal{L} X_A R_i^{\alpha-1} \text{tr} \underline{\chi}) = \widehat{\chi} \cdot \nabla \mathcal{L}_{X_A} \mathcal{L}_{R_i}^{\alpha-1} \widehat{\chi} + \dots, \quad \text{div} \mathcal{L}_{X_A} \mathcal{L}_{R_i}^{\alpha-1} \widehat{\chi} = \mathcal{L} X_A R_i^{\alpha-1} \text{tr} \underline{\chi} + \dots.$$

The new idea is using elliptic estimates and rewriting the right hand side of the transport equations to avoid the loss of derivatives. To rewrite the equation, we need to use the inhomogeneous wave equation satisfied by $\rho := \psi_0^2$:

$$\square_g \rho = \frac{d(\log(c))}{d\rho} g^{\mu\nu} \partial_\mu \rho \partial_\nu \rho + 2g^{\mu\nu} \partial_\mu \psi_0 \partial_\nu \psi_0 \quad (7.1)$$

Then following the same procedure as in [18], we obtain:

$$\underline{L}(\mu \text{tr} \underline{\chi} - \check{f}) = 2\underline{L}\mu \text{tr} \underline{\chi} - \frac{1}{2}\mu(\text{tr} \underline{\chi})^2 - \mu|\widehat{\chi}|^2 + \check{g}, \quad (7.2)$$

with

$$\begin{aligned} \check{f} &= -\frac{1}{2} \frac{d(c^2)}{d\rho} L\rho, \\ \check{g} &= \left(2 \left(\frac{d(c)}{d\rho} \right)^2 + c \frac{d^2(c)}{d\rho^2} \right) (L\rho L\rho - \mu|\check{\rho}|^2) \\ &\quad + 2c \frac{d(c)}{d\rho} \left((L\psi_0 \underline{L}\psi_0 - \mu|\check{\rho}\psi_0|^2) + \left(\frac{1}{4} \frac{\mu|\check{\rho}|^2}{c^2} - \zeta^A \check{\rho}_A \rho \right) \right). \end{aligned}$$

being of order at most 1 and regular as $\mu \rightarrow 0$. So there will be no loss of derivatives by integrating the equation (7.2). Let us introduce the notation F_α :

$$F_\alpha = \mu \not{d} (X_A R_i^{\alpha-1} \text{tr} \underline{\chi}) - \not{d} (X_A R_i^{\alpha-1} \check{f}).$$

Therefore F_α satisfies the equation:

$$\not{L} F_\alpha + (\text{tr} \underline{\chi} - 2\mu^{-1} \underline{L}\mu) F_\alpha = \left(-\frac{1}{2} \text{tr} \underline{\chi} + 2\mu^{-1} \underline{L}\mu \right) \not{d} (X_A R_i^{\alpha-1} \check{f}) - \mu \not{d} (X_A R_i^{\alpha-1} (|\widehat{\chi}|^2)) + g_\alpha, \quad (7.3)$$

with g_α in the following schematic expression (by setting all the numerical constants to be 1):

$$\begin{aligned} g_\alpha &= \not{L}_{X_A} \not{L}_{R_i}^{\alpha-1} g_0 + \sum_{|\beta_1|+|\beta_2|=|\alpha|-1} \not{L}_{\mathcal{R}_i}^{\beta_1} \not{L}_{(\mathcal{R}_i)\underline{Z}} F_{\beta_2} + \sum_{|\beta_1|+|\beta_2|=|\alpha|-1} \not{L}_{\mathcal{R}_i}^{\beta_1} \left((\mu \mathcal{R}_i \text{tr} \underline{\chi} + \mathcal{R}_i \underline{L}\mu + {}^{(\mathcal{R}_i)} \underline{Z}\mu) \not{d} (\mathcal{R}_i^{\beta_2} \text{tr} \underline{\chi}) \right) \\ &\quad + \sum_{|\beta_1|+|\beta_2|=|\alpha|-1} \not{L}_{\mathcal{R}_i}^{\beta_1} \left(\mathcal{R}_i \mu [\not{L}_{\underline{L}} \not{d} (\mathcal{R}_i^{\beta_2} \text{tr} \underline{\chi}) + \text{tr} \underline{\chi} \not{d} (\mathcal{R}_i^{\beta_2} \text{tr} \underline{\chi}) + \not{d} (\mathcal{R}_i^{\beta_2} (|\widehat{\chi}|^2))] + \mathcal{R}_i \text{tr} \underline{\chi} \not{d} (\mathcal{R}_i^{\beta_2} \check{f}) \right). \end{aligned}$$

Here

$$g_0 = \not{d}\check{g} - \frac{1}{2} \text{tr} \underline{\chi} \not{d} (\check{f} - 2\underline{L}\mu) - (\not{d}\mu) (\underline{L} \text{tr} \underline{\chi} + |\widehat{\chi}|^2)$$

and \mathcal{R}_i is either R_i or X_A . Moreover, in the string of \mathcal{R}_i 's, X_A appears once exactly and in front of R_i 's. For any form ξ , since $|\xi| \underline{L} |\xi| = (\xi, \not{L} \xi) - \xi \cdot \widehat{\chi} \cdot \xi - \frac{1}{2} \text{tr} \underline{\chi} |\xi|^2$, we have $|\underline{L} \xi| \leq |\not{L} \xi| + |\widehat{\chi}| |\xi| - \frac{1}{2} \text{tr} \underline{\chi} |\xi|$. Applying these to (7.3), we obtain

$$\underline{L} |F_\alpha| \leq (\mu^{-1} \underline{L}\mu - \frac{3}{2} \text{tr} \underline{\chi} + |\widehat{\chi}|) |F_\alpha| + (2\mu^{-1} |\underline{L}\mu| - \text{tr} \underline{\chi}) |\not{d} X_A R_i^{\alpha-1} \check{f}| + |\mu \not{d} X_A R_i^{\alpha-1} (|\widehat{\chi}|^2)| + |g_\alpha|. \quad (7.4)$$

or

$$\underline{L} |F_\alpha| \leq (\mu^{-1} \underline{L}\mu - \frac{3}{2} \text{tr} \underline{\chi} + |\widehat{\chi}|) |F_\alpha| + (2\mu^{-1} |\underline{L}\mu| - \text{tr} \underline{\chi}) |\not{d} X_A R_i^{\alpha-1} \check{f}| + \widetilde{g}_\alpha. \quad (7.5)$$

with

$$\widetilde{g}_\alpha := |\mu \not{d} X_A R_i^{\alpha-1} (|\widehat{\chi}|^2)| + |g_\alpha|$$

In terms of $\underline{\chi}'_{AB} = \underline{\chi}_{AB} + \frac{\not{g}_{AB}}{\underline{u}-t}$, we write the above inequality in optical coordinates as:

$$\begin{aligned} &\frac{\partial |F_\alpha|}{\partial t} + \frac{3}{t-\underline{u}} |F_\alpha| + \left(-\mu^{-1} \frac{\partial \mu}{\partial t} + \frac{3}{2} \text{tr} \underline{\chi}' - |\widehat{\chi}| \right) |F_\alpha| \\ &\leq (2\mu^{-1} |\partial_t \mu| - \text{tr} \underline{\chi}) |\not{d} X_A R_i^{\alpha-1} \check{f}| + \widetilde{g}_\alpha. \end{aligned} \quad (7.6)$$

Multiplying both sides by $-(t - \underline{u})^3$, we obtain:

$$\begin{aligned} & \partial_t \left((-t + \underline{u})^3 |F_\alpha| \right) + \left(-\mu^{-1} \frac{\partial \mu}{\partial t} + \frac{3}{2} \text{tr} \underline{\chi}' - |\widehat{\chi}| \right) (-t + \underline{u})^3 |F_\alpha| \\ & \leq (-t + \underline{u})^3 (2\mu^{-1} |\partial_t \mu| - \text{tr} \underline{\chi}) |\not{d} X_A R_i^{\alpha-1} \check{f}| + (-t + \underline{u})^3 \widetilde{g}_\alpha. \end{aligned}$$

For the term involving $-\mu^{-1} \partial_t \mu$ on the left hand side, if $\mu < 1/10$, then $\partial_t \mu < 0$, and this term can be dropped. Otherwise, since $\mu \geq 1/10$, $-\mu \partial_t \mu$ can be bounded in absolute value by $C(-t)^{-2}$, with C being an absolute constant. Therefore, in view of the pointwise bounds for $\underline{\chi}'$ and $\widehat{\chi}$, the second term on the left hand side can be bounded, after it is moved to the right hand side, as:

$$C(-t)^{-2} (-t + \underline{u})^3 |F_\alpha|.$$

So this can be treated by Gronwall and we obtain:

$$\begin{aligned} (-t + \underline{u})^3 |F_\alpha(t, \underline{u})| & \leq C \left(\int_{-r_0}^t ((-t' + \underline{u})^3 (2\mu^{-1} |\partial_t \mu| - \text{tr} \underline{\chi})(t, \underline{u}) |\not{d} X_A R_i^{\alpha-1} \check{f}(t', \underline{u})|) dt' \right. \\ & \quad \left. + \int_{-r_0}^t ((-t' + \underline{u})^3 \widetilde{g}_\alpha(t', \underline{u})) dt' \right) \\ & := I + II \end{aligned}$$

Let us now investigate the L^2 norms of the quantities appearing in the above inequality. For a smooth function ϕ , the L^2 norm of it on $[0, \delta] \times \mathbb{S}^2$ is defined by:

$$\|\phi(t)\|_{L^2([0, \delta] \times \mathbb{S}^2)} := \sqrt{\int_{[0, \delta] \times \mathbb{S}^2} \phi(t, \underline{u})^2 r_0^{-2} d\mu_{g(-r_0, 0)} d\underline{u}}$$

On Σ_t^δ , the L^2 norm is defined by:

$$\|\phi(t)\|_{L^2(\Sigma_t^\delta)} := \sqrt{\int_{[0, \delta] \times \mathbb{S}^2} \phi(t, \underline{u})^2 d\mu_{g(t, \underline{u})} d\underline{u}}$$

The relation between these two norms is:

$$(-t) \|\phi(t)\|_{L^2([0, \delta] \times \mathbb{S}^2)} \lesssim \|\phi(t)\|_{L^2(\Sigma_t^\delta)} \lesssim (-t) \|\phi(t)\|_{L^2([0, \delta] \times \mathbb{S}^2)}$$

This discussion also applies to tensors.

Now we give an estimate for $(-t)^2 \|F_\alpha\|_{L^2(\Sigma_t^\delta)}$. In view of

$$(-t)^3 \|F_\alpha(t)\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} \lesssim (-t)^2 \|F_\alpha\|_{L^2(\Sigma_t^\delta)} \lesssim (-t)^3 \|F_\alpha(t)\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)},$$

we only need to give an estimate for $(-t)^3 \|F_\alpha(t)\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)}$. We first estimate I :

$$\|I\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} \lesssim \int_{-r_0}^t (-t')^3 \|(2\mu^{-1} |\partial_t \mu| - \text{tr} \underline{\chi}) |\not{d} X_A R_i^{\alpha-1} \check{f}|(t', \underline{u}')\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} dt'$$

In view of the definition of $\not{d}X_A R_i^{\alpha-1} \check{f}$ and (5.16), we have:

$$\begin{aligned}
& \| \not{d}X_A R_i^{\alpha-1} \check{f}(t', \underline{u}) \|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} \\
& \lesssim \delta^{1/2} (-t')^{-3} \sum_{|\beta| \leq |\alpha|+1} \| T R_i^\beta \psi_0 \|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} \\
& + \delta^{-1/2} (-t')^{-3} \sum_{|\beta| \leq |\alpha|} \| R_i^\beta \psi_0 \|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} \\
& \lesssim \delta^{1/2} (-t')^{-3} \sum_{|\beta| \leq |\alpha|+1} \| T R_i^\beta \psi_0 \|_{L^2([0, \underline{u}] \times \mathbb{S}^2)}
\end{aligned}$$

Let t_0 be the first point in evolution for which $\mu_m(t_0) = \frac{1}{10}$. When $t' \in [-r_0, t_0]$, we have $\mu_m(t') \geq 1/10$. The contribution to $\|I\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)}$ is bounded by:

$$\begin{aligned}
& C \delta^{1/2} \int_{-r_0}^{t_0} (-t')^{-1} \sum_{|\beta| \leq |\alpha|+1} \| T R_i^\beta \psi_0 \|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} dt' \\
& \leq C \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \sqrt{E_{\leq |\alpha|+2}(t', \underline{u})} dt' \\
& \leq C \delta^{1/2} \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}}(t') (-t')^{-2} \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt'
\end{aligned}$$

In order to treat the case when $t' \in [t_0, t]$, we need an analogy to the Lemma 8.1 in [18], concerning the behavior of $(\underline{L}(\log \mu))_-$. Let us define the following quantities:

$$\begin{aligned}
M(t) &:= \max_{(\underline{u}, \theta), (t, \underline{u}, \theta) \in W_{shock}} |(\underline{L}(\log \mu))_-(t, \underline{u}, \theta)|, \\
I_a(t) &:= \int_{t_0}^t \mu_m^{-a}(\tau) M(\tau) d\tau,
\end{aligned}$$

where $a \in \mathbb{R}_{>0}$ is a constant.

Lemma 7.1. (1) Given a constant $a \geq 4$, for all $t \in [t_0, t^*)$, we have

$$I_a(t) \lesssim a^{-1} \mu_m^{-a}(t). \quad (7.7)$$

(2) For $a \geq 4$ and δ sufficiently small, there is an absolute constant C_0 independent of a , so that for all $\tau \in [-r_0, t]$, we have

$$\mu_m^a(t) \leq C_0 \mu_m^a(\tau) \quad (7.8)$$

Proof. (1) By Proposition 4.6, for $t \geq t_0$, the minimum of $r_0^2(\underline{L}\mu)(-r_0, \underline{u}, \theta)$ on $[0, \delta] \times \mathbb{S}^2$ is negative and we denote it by

$$-\eta_m = \min_{(\underline{u}, \theta) \in [0, \delta] \times \mathbb{S}^2} \{r_0^2(\underline{L}\mu)(-r_0, \underline{u}, \theta)\}. \quad (7.9)$$

We notice that $1 \leq \eta_m \leq C_m$ where C_m is a constant depending on the initial data. In view of the asymptotic expansion for $(\underline{L}\mu)(t, \underline{u}, \theta)$ in Lemma 4.4, we have

$$\mu(t, \underline{u}, \theta) = 1 - \left(\frac{1}{t} + \frac{1}{r_0} \right) r_0^2(\underline{L}\mu)(-r_0, \underline{u}, \theta) + O(\delta M^4) \left(\frac{1}{t^2} - \frac{1}{r_0^2} \right). \quad (7.10)$$

We fix an $s \in (t_0, t^*)$ in such a way that $t_0 \leq t < s < t^*$. There exists $(\underline{u}_s, \theta_s) \in [0, \delta] \times \mathbb{S}^2$ and $(\underline{u}_m, \theta_m) \in [0, \delta] \times \mathbb{S}^2$ so that

$$\mu(s, \underline{u}_s, \theta_s) = \mu_m(s), \quad r_0^2(\underline{L}\mu)(-r_0, \underline{u}_m, \theta_m) = -\eta_m. \quad (7.11)$$

We claim that

$$|\eta_m + r_0^2(\underline{L}\mu)(-r_0, \underline{u}_s, \theta_s)| \leq O(\delta M^4). \quad (7.12)$$

Indeed, one can apply (7.10) to $\mu(s, \underline{u}_m, \theta_m)$ and $\mu(s, \underline{u}_s, \theta_s)$ to derive

$$\begin{aligned} \mu(s, \underline{u}_s, \theta_s) &= 1 - \left(\frac{1}{s} + \frac{1}{r_0}\right) (-\eta_m + d_{ms}) + O(\delta M^4) \left(\frac{1}{t^2} - \frac{1}{r_0^2}\right) \\ \mu(s, \underline{u}_m, \theta_m) &= 1 - \left(\frac{1}{s} + \frac{1}{r_0}\right) (-\eta_m) + O(\delta M^4) \left(\frac{1}{t^2} - \frac{1}{r_0^2}\right), \end{aligned} \quad (7.13)$$

where the quantity $d_{ms} > 0$ is defined as

$$d_{ms} := \eta_m + r_0^2(\underline{L}\mu)(-r_0, \underline{u}_s, \theta_s). \quad (7.14)$$

Since $\mu(s, \underline{u}_s, \theta_s) \leq \mu(s, \underline{u}_m, \theta_m)$, we have

$$0 < -\left(\frac{1}{s} + \frac{1}{r_0}\right) d_{ms} \leq O(\delta M^4) \left(\frac{1}{t^2} - \frac{1}{r_0^2}\right). \quad (7.15)$$

Hence,

$$-\left(\frac{1}{t} + \frac{1}{r_0}\right) d_{ms} \leq -\left(\frac{1}{s} + \frac{1}{r_0}\right) d_{ms} \leq O(\delta M^4) \left(\frac{1}{t^2} - \frac{1}{r_0^2}\right),$$

which implies

$$d_{ms} \leq O(\delta M^4) \left(\frac{1}{r_0} - \frac{1}{t}\right) \leq O(\delta M^4). \quad (7.16)$$

With this preparation, one can derive precise upper and lower bounds for $\mu_m(t)$.

We pick up a $(\underline{u}'_m, \theta'_m) \in [0, \delta] \times \mathbb{S}^2$ in such a way that $\mu(t, \underline{u}'_m, \theta'_m) = \mu_m(t)$. For the lower bound, by virtue of Lemma 4.4, we have

$$\begin{aligned} \mu_m(t) &= \mu(t, \underline{u}'_m, \theta'_m) = \mu(s, \underline{u}'_m, \theta'_m) + \int_s^t (\underline{L}\mu)(t', \underline{u}'_m, \theta'_m) dt' \\ &\geq \mu_m(s) + \int_s^t \frac{\eta_m}{-t'^2} + \frac{O(\delta M^4)}{(-t')^3} dt' \\ &\geq \mu_m(s) + \left(\eta_m - \frac{1}{2a}\right) \left(\frac{1}{t} - \frac{1}{s}\right). \end{aligned} \quad (7.17)$$

In the last step, we take sufficiently small δ so that $O(\delta M^4) \leq \frac{1}{2a}$.

For the upper bound, in view of Lemma 4.4 and (7.16), we have

$$\begin{aligned} \mu_m(t) &\leq \mu(t, \underline{u}_s, \theta_s) = \mu_m(s) + \int_s^t (\underline{L}\mu)(t', \underline{u}_s, \theta_s) dt' \\ &= \mu_m(s) + \int_s^t \frac{\eta_m - d_{ms}}{-t'^2} + \frac{O(\delta M^4)}{(-t')^3} dt' \\ &\leq \mu_m(s) + \int_s^t \frac{\eta_m}{-t'^2} + \frac{O(\delta M^4)}{t'^2} dt' \\ &\leq \mu_m(s) + \left(\eta_m + \frac{1}{2a}\right) \left(\frac{1}{t} - \frac{1}{s}\right) \end{aligned} \quad (7.18)$$

In the last step, we also take sufficiently small δ so that $O(\delta M^4) \leq \frac{1}{2a}$.

For $I_a(t)$, first of all, we have

$$\begin{aligned} I_a(t) &\lesssim \int_{t_0}^t \left(\mu_m(s) + \left(\eta_m - \frac{1}{2a} \right) \left(\frac{1}{t} - \frac{1}{s} \right) \right)^{-a-1} t'^{-2} dt' \\ &= \int_{\tau}^{\tau_0} \left(\mu_m(s) + \left(\eta_m - \frac{1}{2a} \right) (\tau - \tau_s) \right)^{-a-1} d\tau' \\ &\leq \frac{1}{\eta_m - \frac{1}{2a}} \frac{1}{a} \left(\mu_m(s) + \left(\eta_m - \frac{1}{2a} \right) (\tau - \tau_s) \right)^{-a}. \end{aligned}$$

Hence,

$$\begin{aligned} I_a(t) &\lesssim \frac{1}{a} \left(\mu_m(s) + \left(\eta_m - \frac{1}{2a} \right) \left(\frac{1}{t} - \frac{1}{s} \right) \right)^{-a} \\ &\leq \frac{1}{a} \frac{(\mu_m(s) + (\eta_m - \frac{1}{2a}) (\frac{1}{t} - \frac{1}{s}))^{-a}}{(\mu_m(s) + (\eta_m + \frac{1}{2a}) (\frac{1}{t} - \frac{1}{s}))^{-a}} \mu_m^{-a}(t) \\ &\leq \frac{1}{a} \frac{((\eta_m - \frac{1}{2a}) (\frac{1}{t} - \frac{1}{s}))^{-a}}{((\eta_m + \frac{1}{2a}) (\frac{1}{t} - \frac{1}{s}))^{-a}} \mu_m^{-a}(t). \end{aligned} \tag{7.19}$$

Since as $a \rightarrow \infty$, one has

$$\frac{(\eta_m - \frac{1}{2a})^{-a}}{(\eta_m + \frac{1}{2a})^{-a}} \rightarrow e^{\frac{1}{\eta_m}}.$$

The limit is an absolute constant. Therefore, (7.19) yields the proof for part (1) of the lemma. The proof for part (1') is exactly the same.

(2) We start with an easy observation: if $\mu(t, \underline{u}, \theta) \leq 1 - \frac{1}{a}$, then $\underline{L}\mu(t, \underline{u}, \theta) \lesssim -a^{-1}$. In fact, we claim that $(\frac{1}{t} + \frac{1}{r_0})r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) \geq \frac{1}{2}a^{-1}$. Otherwise, for sufficiently small δ (say $\delta^{1/4} \leq a^{-1}$), according to the expansion for $\mu(t, \underline{u}, \theta)$, i.e. $\mu(t, \underline{u}, \theta) = \mu(-r_0, \underline{u}, \theta) - (\frac{1}{t} + \frac{1}{r_0})r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) + \frac{O(\delta)}{t}$, we have

$$\mu(t, \underline{u}, \theta) > 1 - \frac{1}{2a} - C \frac{\delta}{t} \geq 1 - \frac{1}{a}.$$

which is a contradiction. So in view of the fact that $\frac{tr_0}{t+r_0}$ is bounded above by a negative absolute constant when $t \in (-r_0, -1]$, we have $r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) \lesssim -a^{-1}$. Therefore the expansion of $\underline{L}\mu$ implies $\underline{L}\mu(t, \underline{u}, \theta) \lesssim -a^{-1}t^{-2}$. In particular, this observation implies that, if there is a $t' \in [-r_0, s_m^*]$, so that $\mu_m(t') \leq 1 - a^{-1}$, then for all $t \geq t'$, we have $\mu_m(t) \leq 1 - a^{-1}$. This allows us to define a time t_1 , such that it is the minimum of all such t' with $\mu_m(t') \leq 1 - a^{-1}$.

We now prove the lemma. If $\tau \leq t_1$, since $\mu_m(t) \leq 1$, we have

$$\mu_m^{-a}(\tau) \leq (1 - \frac{1}{a})^{-a} \leq C_0 \leq C_0 \mu_m^{-a}(t).$$

If $\tau \geq t_1$, then $\mu_m(\tau) \leq 1 - \frac{1}{a}$. let $\mu_m(\tau) = \mu(\tau, \underline{u}_\tau, \theta_\tau)$. We know that $\mu(t, \underline{u}_\tau, \theta_\tau)$ is decreasing in t for $t \geq \tau$. Therefore, we have

$$\mu_m(t) \leq \mu(t, \underline{u}_\tau, \theta_\tau) \leq \mu(\tau, \underline{u}_\tau, \theta_\tau) = \mu_m(\tau).$$

The proof now is complete. □

Let us continue to estimate $\|I\|_{L^2([0,\underline{u}]\times\mathbb{S}^2)}$. When $t' \in [t_0, t]$, the contribution of the term $\mu^{-1}\partial_t\mu$ is bounded through Lemma 7.1 by:

$$C\delta^{1/2} \int_{t_0}^t \mu_m^{-b_{|\alpha|+2}}(t')(-t')^{-1}M(t')\sqrt{\tilde{E}_{\leq|\alpha|+2}(t',\underline{u})}dt' \leq \frac{C\delta^{1/2}(-t)^{-1}}{b_{|\alpha|+2}}\mu_m^{-b_{|\alpha|+2}}(t)\sqrt{\tilde{E}_{\leq|\alpha|+2}(t,\underline{u})}$$

Therefore we have the following estimates for I :

$$\begin{aligned} \|I\|_{L^2([0,\underline{u}]\times\mathbb{S}^2)} &\leq \frac{C\delta^{1/2}(-t)^{-1}}{b_{|\alpha|+2}}\mu_m^{-b_{|\alpha|+2}}(t)\sqrt{\tilde{E}_{\leq|\alpha|+2}(t,\underline{u})} \\ &\quad + C\delta^{1/2} \int_{-r_0}^t (-t')^{-2}\mu_m^{-b_{|\alpha|+2}}(t')\sqrt{\tilde{E}_{\leq|\alpha|+2}(t',\underline{u})}dt'. \end{aligned} \quad (7.20)$$

Next we move to the estimate for $\|II\|_{l^2([0,\underline{u}]\times\mathbb{S}^2)}$. We first investigate the structure of g_α , which is the sum of four terms. In order to estimate the first term, $\mathcal{L}_{R_i}^\alpha g_0$, we rewrite g_0 in another way. Taking the trace of the equation (2.35), we can rewrite the last term in g_0 as

$$-(\not{d}\mu) (\underline{L}\text{tr}\chi + |\chi|^2) = -(\not{d}\mu) (\text{etr}\chi - \text{tr}\alpha').$$

Therefore the contribution of last term in g_0 to $\|\mathcal{L}_{R_i}^\alpha g_0\|_{L^2([0,\delta]\times\mathbb{S}^2)}$ is $\|\mathcal{L}_{R_i}^\alpha(\not{d}\mu) (\text{etr}\chi - \text{tr}\alpha')\|_{L^2([0,\delta]\times\mathbb{S}^2)}$. Since the top order is $|\alpha| + 2$, there is no top order optical terms in this contribution. In regard to the L^2 norms of lower order optical terms, we introduce the following notations:

$$\begin{aligned} \mathcal{A}_l^u(t) &:= \max_{|\alpha|=l} \|\mathcal{L}_{R_i}^\alpha \chi\|_{L^2(\Sigma_t^u)}, \\ \mathcal{A}_l'^u(t) &:= \max_{|\alpha|=l} \|\mathcal{L}_{R_i}^\alpha \chi'\|_{L^2(\Sigma_t^u)}, \\ \mathcal{B}_{m,l+1}^u(t) &:= \max_{|\beta|+|\gamma|=l} \|R_i^{\beta+1} T^\gamma \mu\|_{L^2(\Sigma_t^u)}, \\ \mathcal{A}_{\leq l}^u(t) &:= \sum_{k=1}^l \mathcal{A}_k^u(t), \\ \mathcal{A}_{\leq l}'^u(t) &:= \sum_{k=1}^l \mathcal{A}_k'^u(t), \\ \mathcal{B}_{\leq m, \leq l+1}^u(t) &:= \sum_{m' \leq m, l' \leq l} \mathcal{B}_{m', l'+1}^u(t). \end{aligned} \quad (7.21)$$

It's obvious that

$$\|\mathcal{R}_i^\alpha \text{tr}\chi\|_{L^2([0,\delta]\times\mathbb{S}^2)} = \|\mathcal{R}_i^\alpha \text{tr}\chi'\|_{L^2([0,\delta]\times\mathbb{S}^2)} \quad \text{for } |\alpha| \geq 1.$$

To estimate the factor $\|\mathcal{L}_{R_i}^\alpha ((\not{d}\mu)(\text{etr}\chi))\|_{L^2(\Sigma_t^u)}$, we need to use the Leibniz rule. If more than half of the derivatives hit e , then the corresponding contribution is bounded by:

$$\begin{aligned} &C\delta^{1/2}(-t)^{-5} \sum_{|\beta| \leq |\alpha|} \left(\|R_i^\beta Q\psi_0\|_{L^2(\Sigma_t^u)} + \|R_i^\beta \psi_0\|_{L^2(\Sigma_t^u)} \right) \\ &\leq C\delta^{3/2}(-t)^{-5} \sqrt{E_{\leq|\alpha|+2}(t,\underline{u})} \\ &\leq C\delta^{3/2}(-t)^{-5} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t,\underline{u})} \end{aligned}$$

in view of the pointwise estimates of $\not{d}R_i^\beta \mu$, $R_i^\beta \text{tr}\underline{\chi}$ and $R_i^\beta \psi$, $R_i^\beta Q\psi$ when $|\beta| \leq N_\infty$. Similarly, if more than half of the derivatives hit $\text{tr}\underline{\chi}$ and $\not{d}\mu$, their corresponding contribution are bounded by:

$$\delta(-t)^{-5} \mathcal{A}'_{\leq|\alpha|}(t) \quad \text{and} \quad \delta(-t)^{-5} \mathcal{B}_{0,\leq|\alpha|+1}(t)$$

respectively.

On the other hand, in view of Proposition 6.3 and 6.4, we have the following estimates for $\mathcal{A}'_{\leq|\alpha|}(t)$ and $\mathcal{B}_{0,\leq|\alpha|+1}(t)$:

$$\begin{aligned} \mathcal{A}'_{\leq|\alpha|}(t) &\lesssim \delta^{1/2} \int_{-r_0}^t (-t')^{-3} \mu_m^{-1/2}(t') \sqrt{E_{\leq|\alpha|+2}(t', \underline{u})} dt', \\ &\lesssim \delta^{1/2} (-t)^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})}; \\ (-t)^{-1} \mathcal{B}_{0,\leq|\alpha|+1}(t) &\lesssim (-r_0)^{-1} \mathcal{B}_{0,\leq|\alpha|+1}(-r_0) \\ &\quad + \delta^{1/2} \int_{-r_0}^t (-\tau)^{-2} \left(\sqrt{E_{\leq|\alpha|+2}(\tau, \underline{u})} + \mu_m^{-1/2}(\tau) (-\tau)^{-1} \sqrt{E_{\leq|\alpha|+2}(\tau, \underline{u})} \right) d\tau, \\ &\lesssim (-r_0)^{-1} \mathcal{B}_{0,\leq|\alpha|+1}(-r_0) \\ &\quad + (-t)^{-1} \left(\mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} + \mu_m^{-b_{|\alpha|+2}-1/2}(t) (-t)^{-1} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \right). \end{aligned}$$

Combining the above estimates, we have:

$$\begin{aligned} \|\mathcal{L}_{\mathcal{R}_i}^\alpha((\not{d}\mu)(\text{etr}\underline{\chi}))\|_{L^2(\Sigma_t^{\underline{u}})} &\lesssim \delta(-t)^{-5} \mathcal{B}_{0,\leq|\alpha|+1}(-r_0) \\ &\quad + C\delta^{3/2}(-t)^{-5} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\ &\quad + C\delta^{3/2}(-t)^{-6} \mu_m^{-b_{|\alpha|+2}+1/2}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})}. \end{aligned} \tag{7.22}$$

For the last term in $\mathcal{L}_{\mathcal{R}_i}^\alpha g_0$, we are still left with the term $\|\mathcal{L}_{\mathcal{R}_i}^\alpha((\not{d}\mu) \text{tr}\alpha')\|_{L^2(\Sigma_t^{\underline{u}})}$. If more than half of the derivatives hit $\not{d}\mu$, the estimates are exactly the same as the previous case. If more than half of the derivatives hit $\text{tr}\alpha'$, then the contribution is bounded by:

$$\delta^{1/2}(-t)^{-4} \|\not{d}R_i^{\alpha+1}\psi_0\|_{L^2(\Sigma_t^{\underline{u}})} \lesssim \delta^{1/2}(-t)^{-5} \mu_m^{-1/2}(t) \sqrt{E_{\leq|\alpha|+2}(t, \underline{u})} \tag{7.23}$$

Combining (7.22) and (7.23), we have

$$\begin{aligned} \|\mathcal{L}_{\mathcal{R}_i}^\alpha((\not{d}\mu)(\text{etr}\underline{\chi} - \text{tr}\alpha'))\|_{L^2(\Sigma_t^{\underline{u}})} &\lesssim \delta(-t)^{-4} \mathcal{B}_{0,\leq|\alpha|+1}(-r_0) \\ &\quad + C\delta^{3/2}(-t)^{-4} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\ &\quad + C\delta^{1/2}(-t)^{-5} \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})}. \end{aligned} \tag{7.24}$$

Next we consider the contribution from the second term in the expression of g_0 , whose L^2 norm is given by $\|\mathcal{L}_{\mathcal{R}_i}^\alpha(\text{tr}\underline{\chi} \cdot \not{d}(\check{f} - 2L\underline{\mu}))\|_{L^2(\Sigma_t^{\underline{u}})}$. For the first term $\|\mathcal{L}_{\mathcal{R}_i}^\alpha(\text{tr}\underline{\chi} \cdot \not{d}\check{f})\|_{L^2(\Sigma_t^{\underline{u}})}$, if more than half of the derivatives hit $\text{tr}\underline{\chi}$, then it is bounded by

$$(-t)^{-3} \mathcal{A}'_{\leq|\alpha|}(t) \lesssim \delta^{1/2}(-t)^{-5} \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})}. \tag{7.25}$$

If more than half of the derivatives hit $\not{d}\check{f}$, then this term is bounded by:

$$\delta^{1/2}(-t)^{-3} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \lesssim \delta^{1/2}(-t)^{-3} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})}. \tag{7.26}$$

Here we also used (5.16) if most of the derivatives hit ψ_0 instead of $L\psi_0$. The treatment for the second term $\|\mathcal{L}_{\mathcal{R}_i}^\alpha (\text{tr}\underline{\chi} \cdot \mathcal{L}\underline{L}\mu)\|_{L^2(\Sigma_t^\#)}$ is similar. If more than half of the derivatives hit $\text{tr}\underline{\chi}$, the estimate is exactly the same as (7.25). If more than half of the derivatives hit $\mathcal{L}\underline{L}\mu$, then this term is bounded by

$$\delta^{1/2}(-t)^{-3}\mu_m^{-b_{|\alpha|+2}}(t)\sqrt{\tilde{E}_{\leq|\alpha|+2}(t,\underline{u})} + \delta^{1/2}(-t)^{-4}\mu_m^{-b_{|\alpha|+2}-1/2}(t)\sqrt{\underline{E}_{\leq|\alpha|+2}(t,\underline{u})}. \quad (7.27)$$

Combining (7.25), (7.26) and (7.27), we have:

$$\begin{aligned} & \|\mathcal{L}_{\mathcal{R}_i}^\alpha (\text{tr}\underline{\chi} \cdot \mathcal{L}(\check{f} - 2\underline{L}\mu))\|_{L^2(\Sigma_t^\#)} \\ & \lesssim \delta^{1/2}(-t)^{-3}\mu_m^{-b_{|\alpha|+2}}(t)\sqrt{\tilde{E}_{\leq|\alpha|+2}(t,\underline{u})} + \delta^{1/2}(-t)^{-4}\mu_m^{-b_{|\alpha|+2}-1/2}(t)\sqrt{\underline{E}_{\leq|\alpha|+2}(t,\underline{u})}. \end{aligned} \quad (7.28)$$

To complete the estimates for the contribution from $\mathcal{L}_{R_i}^\alpha g_0$, we finally estimate the contribution from \check{g} : $\|\mathcal{L}_{R_i}^\alpha (\mathcal{L}\check{g})\|_{L^2(\Sigma_t^\#)}$. Recall the expression of \check{g} :

$$\begin{aligned} \check{g} = & \frac{1}{2} \frac{d^2(c^2)}{d\rho^2} [\underline{L}\rho L\rho - \mu|\mathcal{L}\rho|^2] - \frac{1}{2} \frac{d(c^2)}{d\rho} \Omega^{-1} \frac{d\Omega}{d\rho} [\underline{L}\rho L\rho - \mu|\mathcal{L}\rho|^2] \\ & + \frac{d(c^2)}{d\rho} [L\psi_0 \underline{L}\psi_0 - \mu|\mathcal{L}\psi_0|^2] + \frac{d(c^2)}{d\rho} \left[\frac{1}{4} c^{-2} \mu|\mathcal{L}\rho|^2 - \zeta^A \cdot \mathcal{L}_A \rho \right]. \end{aligned}$$

Here we will only treat the top order terms, namely, when all the derivatives hit one factor. The estimates for other lower order terms follow in the same manner. We first consider the contribution from the term $\underline{L}\rho L\rho$. When all of $\mathcal{L}_{\mathcal{R}_i}^\alpha$ hit $L\rho$, the contribution is bounded by:

$$\delta^{3/2}(-t)^{-4}\sqrt{\underline{E}_{\leq|\alpha|+2}(t,\underline{u})} \leq \delta^{3/2}(-t)^{-4}\mu_m^{-b_{|\alpha|+2}}(t)\sqrt{\tilde{E}_{\leq|\alpha|+2}(t,\underline{u})}. \quad (7.29)$$

Here we also used (5.16) when $\mathcal{L}_{\mathcal{R}_i}^\alpha$ hit ψ_0 .

When all of $\mathcal{L}_{R_i}^\alpha$ hit $\underline{L}\rho$, the contribution is bounded by:

$$\begin{aligned} \delta^{1/2}(-t)^{-3} \|\underline{L}\mathcal{R}_i^{\alpha+1}\psi_0\|_{L^2(\Sigma_t^\#)} & \lesssim \delta^{1/2}(-t)^{-4} \|\underline{L}R_i^{\alpha+1}\psi_0 + \nu R_i^{\alpha+1}\psi_0\|_{L^2(\Sigma_t^\#)} \\ & \quad + \delta^{3/2}(-t)^{-5} \sqrt{\underline{E}_{\leq|\alpha|+2}(t,\underline{u})} \\ & \lesssim \delta^{1/2}(-t)^{-5} \mu_m^{-1/2}(t) \sqrt{\underline{E}_{\leq|\alpha|+2}(t,\underline{u})} \\ & \quad + \delta^{3/2}(-t)^{-5} \sqrt{\underline{E}_{\leq|\alpha|+2}(t,\underline{u})} \\ & \lesssim \delta^{1/2}(-t)^{-5} \mu_m^{-1/2-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t,\underline{u})} \\ & \quad + \delta^{3/2}(-t)^{-5} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t,\underline{u})}. \end{aligned} \quad (7.30)$$

Here (5.16) is used when $\mathcal{L}_{\mathcal{R}_i}^\alpha$ hit ψ_0 .

Next we consider the contribution of $\mu|\mathcal{L}\rho|^2$. This is bounded by

$$\delta^{3/2}(-t)^{-4} \sqrt{\underline{E}_{\leq|\alpha|+2}(t,\underline{u})} \lesssim \delta^{3/2}(-t)^{-4} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t,\underline{u})}. \quad (7.31)$$

By the definition of ζ , the contribution from $\zeta^A \cdot \mathcal{L}_A \rho$ enjoys the same bound.

Now we move to the contribution from $\mu|\mathcal{L}\psi_0|^2$. This is bounded by:

$$\delta^{1/2}(-t)^{-4} \sqrt{\underline{E}_{\leq|\alpha|+2}(t,\underline{u})} \lesssim \delta^{1/2}(-t)^{-4} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t,\underline{u})}. \quad (7.32)$$

Finally we estimate the most difficult term: $\not{d}(\underline{L}\psi_0 L\psi_0)$. When all of $\not{L}_{\mathcal{R}_i}^\alpha$ and \not{d} hit $L\psi_0$, this contribution is bounded by:

$$\delta^{1/2}(-t)^{-3}\sqrt{E_{\leq|\alpha|+2}(t,\underline{u})} \lesssim \delta^{1/2}(-t)^{-3}\mu_m^{-b_{|\alpha|+2}}(t)\sqrt{\widetilde{E}_{\leq|\alpha|+2}(t,\underline{u})}. \quad (7.33)$$

When $\not{L}_{\mathcal{R}_i}^\alpha$ and \not{d} hit $\underline{L}\psi_0$, the treatment is different. The spacetime integral we want to estimate is (Remember $R_i \sim (-t)\not{d}, R_i \sim (-t)X_A$):

$$\int_{-r_0}^t (-t') \|L\psi_0 \underline{L}\mathcal{R}_i^{\alpha+1}\psi_0\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \leq C\delta^{-1/2} \int_{-r_0}^t (-t')^{-1} \|\underline{L}R_i^{\alpha+1}\psi_0\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt'.$$

In order to use the flux $\underline{F}(t,\underline{u})$, the right hand side of the above is bounded through (5.16) by:

$$\delta^{-1/2} \int_{-r_0}^t (-t')^{-1} \|\underline{L}R_i^{\alpha+1}\psi_0 + \nu R_i^{\alpha+1}\psi_0\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' + \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \sqrt{E_{\leq|\alpha|+2}(t',\underline{u})} dt'.$$

The second term above is bounded in the same way as the contribution of (7.33) to the spacetime integral. We focus on the first term above. By using Hölder inequality, this term is bounded by:

$$\begin{aligned} & \delta^{-1/2} \left(\int_{-r_0}^t (-t')^{-3} dt' \right)^{1/2} \left(\int_{-r_0}^t (-t')^2 \int_0^{\underline{u}} \int_{\mathbb{S}^2} (\underline{L}R_i^{\alpha+1}\psi_0 + \nu R_i^{\alpha+1}\psi_0)^2 d\mu_g d\underline{u}' dt' \right)^{1/2} \\ & \lesssim \delta^{-1/2} (-t)^{-1} \left(\int_0^{\underline{u}} \underline{F}_{\leq|\alpha|+2}(t,\underline{u}') d\underline{u}' \right)^{1/2}. \end{aligned} \quad (7.34)$$

Now we summarize our estimates for the contribution of the first term in $\not{L}_{\mathcal{R}_i}^\alpha g_0$. The L^2 norm of II on $\Sigma_t^{\underline{u}}$ is bounded as:

$$\|II\|_{L^2([0,\underline{u}]\times\mathbb{S}^2)} \leq \int_{-r_0}^t (-t')^3 \|\widetilde{g}_\alpha(t',\underline{u})\|_{L^2([0,\underline{u}]\times\mathbb{S}^2)} dt' \lesssim \int_{-r_0}^t (-t')^2 \|\widetilde{g}_\alpha\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt'. \quad (7.35)$$

Combining (7.29)-(7.34) and (7.28), (7.24), the contribution of $\not{L}_{\mathcal{R}_i}^\alpha g_0$ to right hand side of above is bounded by:

$$\begin{aligned} & \delta \int_{-r_0}^t (-t')^{-2} \mathcal{B}_{0,\leq|\alpha|+1}(-r_0) dt' + \delta^{-1/2} (-t)^{-1} \left(\int_0^{\underline{u}} \underline{F}_{\leq|\alpha|+2}(t,\underline{u}') d\underline{u}' \right)^{1/2} + \\ & \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \left(\mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t',\underline{u})} + \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t',\underline{u})} \right) dt'. \end{aligned} \quad (7.36)$$

Next we consider the second term in g_α , which is a sum of the terms as follows:

$$\not{L}_{\mathcal{R}_i}^\beta \not{L}_{(\mathcal{R}_i)} \underline{F}_{\alpha-\beta-1},$$

which can be systematically rewritten as:

$$\not{L}_{(\mathcal{R}_i)} \underline{F}_{\alpha-1} + \sum_{\beta} \not{L}_{[\not{L}_{\mathcal{R}_i}^\beta, (\mathcal{R}_i)]} \underline{F}_{\alpha-\beta-1}.$$

Here the terms in the sum on the right hand side of the above, which are of lower order, come from the commutator $[\not{L}_{\mathcal{R}_j}, \not{L}_{(\mathcal{R}_i)} \underline{F}]$. The first term, which is of principal order, can be bounded as follows (see [5] and [8])

$$\begin{aligned} |\not{L}_{(\mathcal{R}_i)} \underline{F}_{\alpha-1}| & \leq C(-t)^{-1} \left(|(R_i) \underline{F}| |F_\alpha| + |\not{d} X_A R_k^{\alpha-1} \check{f}| + |R_i \mu| |\not{d}(X_A R_k^{\alpha-2} \text{tr} \chi)| \right. \\ & \quad \left. + |\not{L}_{R_j}^{(R_i)} \underline{F}| (|F_{\alpha-1}| + |\not{d} X_A R_k^{\alpha-2} \check{f}|) \right). \end{aligned}$$

The first term on the right hand side is in terms of the unknown F_α . In view of the estimate for $^{(R_i)}\underline{Z}$:

$$|^{(R_i)}\underline{Z}| \lesssim \delta(-t)^{-2},$$

the contribution of this term can be bounded by Gronwall. While the contribution of the second term is bounded by:

$$\delta^{1/2} \int_{-r_0}^t (-t')^{-2} \sqrt{E_{\leq|\alpha|+2}(t', \underline{u})} dt' \lesssim \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt'. \quad (7.37)$$

For the third term of the right hand side, we employ the estimates for $\mathcal{A}'_{\leq|\alpha|}(t)$ to obtain the contribution of this term is bounded by:

$$\delta^{1/2} \int_{-r_0}^t (-t')^{-3} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt'. \quad (7.38)$$

By Proposition 6.1, we have $|\mathcal{L}_{R_j}^{(R_i)}\underline{Z}| \lesssim \delta(-t)^{-2}$, $|\mathcal{L}_{X_A}^{(R_i)}\underline{Z}| \lesssim \delta(-t)^{-3}$, $|\mathcal{L}_{R_j}^{(X_A)}\underline{Z}| \lesssim \delta(-t)^{-3}$. Therefore the other two terms in the pointwise estimate for $|\mathcal{L}_{(\mathcal{R}_i)}\underline{Z} F_{\alpha-1}|$ are of lower order. So the contribution of the second term in g_α to $\|II\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)}$ is bounded by:

$$\delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' + \delta^{1/2} \int_{-r_0}^t (-t')^{-3} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt'. \quad (7.39)$$

Now we move to the third term in g_α . Again, using Leibniz Rule, the L^2 -norm of this term on $\Sigma_t^{\underline{u}}$ is bounded by:

$$(-t)^{-3} \mathcal{A}'_{\leq|\alpha|}(t) + \delta(-t)^{-5} \sqrt{E_{\leq|\alpha|+2}(t, \underline{u})} + \delta(-t)^{-6} \mathcal{B}_{0, \leq|\alpha|+1}(t).$$

The first term is bounded in the same manner as (7.38) and the rest terms are of lower order, in view of Proposition 6.3 and 6.4. The last term in g_α has a similar structure and can be treated in the same way.

Finally we need to consider the contribution from the term $\mu \mathcal{L}^{(R_i)}[\hat{\chi}]^2$. If not all $\mathcal{L}_{R_i}^\alpha$ hit one single $\hat{\chi}$, then the estimates for $\mathcal{A}'_{\leq|\alpha|}(t)$ implies the estimates of the contribution to II , which is similar to those of the second term in g_α . If all $\mathcal{L}_{R_i}^\alpha$ hit one single $\hat{\chi}$, we need to use the elliptic system for $\hat{\chi}$:

$$\mathcal{L}^* \hat{\chi} = \frac{1}{2} \mathcal{L}(\text{tr} \chi) + \mu^{-1} \left(\zeta \cdot \chi - \frac{1}{2} \text{tr} \chi \zeta \right).$$

The elliptic estimate (3.44) implies:

$$\|\mu |\widehat{\mathcal{L}}_{\mathcal{R}_i}^\alpha \hat{\chi}|\|_{L^2(\Sigma_t^{\underline{u}})} \leq C \|\mu \mathcal{L}(X_A R_i^{\alpha-1} \text{tr} \chi)\|_{L^2(\Sigma_t^{\underline{u}})} + \|\mu |H_\alpha|\|_{L^2(\Sigma_t^{\underline{u}})} + \|\mathcal{L} \mu\|_{L^\infty(\Sigma_t^{\underline{u}})} \|\widehat{\mathcal{L}}_{\mathcal{R}_i}^\alpha \hat{\chi}\|_{L^2(\Sigma_t^{\underline{u}})}. \quad (7.40)$$

Here $\widehat{\mathcal{L}}_{\mathcal{R}_i}$ is the traceless part of $\mathcal{L}_{\mathcal{R}_i}$. H_α is given by

$$\begin{aligned} H_\alpha &= (\mathcal{L}_{\mathcal{R}_i} + \frac{1}{2} \text{tr}^{(\mathcal{R}_i)} \mathcal{L})^\alpha (\mu^{-1} \zeta \cdot \hat{\chi} - \frac{1}{2} \mu^{-1} \zeta \text{tr} \chi) \\ &+ \sum_{|\beta_1|+|\beta_2|=|\alpha|-1} (\mathcal{L}_{\mathcal{R}_i} + \frac{1}{2} \text{tr}^{(\mathcal{R}_i)} \mathcal{L})^{\beta_1} \left(\text{tr}^{(\mathcal{R}_i)} \mathcal{L} \cdot \mathcal{L}^{(\mathcal{R}_i, \beta_2)} \text{tr} \chi + (\mathcal{L}^{(\mathcal{R}_i)} \widehat{\mathcal{L}}) \cdot \widehat{\mathcal{L}}_{\mathcal{R}_i}^{\beta_2} \hat{\chi} \right) \\ &+ \sum_{|\beta_1|+|\beta_2|=|\alpha|-1} (\mathcal{L}_{\mathcal{R}_i} + \frac{1}{2} \text{tr}^{(\mathcal{R}_i)} \mathcal{L})^{\beta_1} \left((\mathcal{R}_i) \widehat{\mathcal{L}} \cdot \nabla \widehat{\mathcal{L}}_{\mathcal{R}_i}^{\beta_2} \hat{\chi} \right). \end{aligned}$$

Using the estimates:

$$|\mu^{-1} \zeta| \lesssim \delta(-t)^{-3}, \quad |\text{tr}^{(\mathcal{R}_i)} \mathcal{L}| \lesssim \delta(-t)^{-2}, \quad |\chi'| \lesssim \delta(-t)^{-3},$$

the right hand side of (7.40) is bounded by:

$$\|F_\alpha\|_{L^2(\Sigma_t^u)} + \delta^{1/2}(-t)^{-2}\mu_m^{-b_{|\alpha|+2}}(t)\sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} + \delta^{3/2}(-t)^{-3}\mu_m^{-b_{|\alpha|+2}-1/2}(t)\sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})}.$$

Therefore we have:

$$\begin{aligned} \|\mu d(X_A R_i^{\alpha-1}|\hat{\chi}|^2)\|_{L^2(\Sigma_t^u)} &\lesssim \delta(-t)^{-3}\|F_\alpha\|_{L^2(\Sigma_t^u)} \\ &\quad + \delta^{3/2}(-t)^{-5}\mu_m^{-b_{|\alpha|+2}}(t)\sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\ &\quad + \delta^{3/2}(-t)^{-6}\mu_m^{-b_{|\alpha|+2}+1/2}(t)\sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \end{aligned}$$

The contribution to II of the first term on the right hand side is bounded by using Gronwall. The contributions from the rest two terms are bounded as:

$$\begin{aligned} &\delta^{3/2} \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}}(t')(-t')^{-3}\sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\ &+ \delta^{3/2} \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t')(-t')^{-4}\sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt'. \end{aligned} \tag{7.41}$$

Combining (7.36), (7.39) and (7.41), we have:

$$\begin{aligned} \|II\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} &\leq \int_{-r_0}^t (-t')^3 \|\tilde{g}_\alpha(t', \underline{u})\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} dt' \leq \int_{-r_0}^t (-t')^2 \|\tilde{g}_\alpha\|_{L^2(\Sigma_{t'}^u)} dt' \\ &\lesssim \delta \int_{-r_0}^t (-t')^{-2} \mathcal{B}_{0, \leq |\alpha|+1}(-r_0) dt' \\ &\quad + \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\ &\quad + \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\ &\quad + \delta^{-1/2}(-t)^{-1} \mu_m^{-b_{|\alpha|+2}}(t) \left(\int_0^{\underline{u}} \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \right)^{1/2}. \end{aligned} \tag{7.42}$$

Now we are ready to give the final estimate for $(-t)^2\|F_\alpha(t, \underline{u})\|_{L^2(\Sigma_t^u)} \sim (-t)^3\|F_\alpha(t, \underline{u})\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)}$. Combining (7.20) and (7.42), we have:

$$\begin{aligned} (-t)^3\|F_\alpha(t, \underline{u})\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} &\leq C\delta \int_{-r_0}^t (-t')^{-2} \mathcal{B}_{0, \leq |\alpha|+1}(-r_0) dt' \\ &\quad + \frac{C\delta^{1/2}(-t)^{-1}}{b_{|\alpha|+2}} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\ &\quad + C\delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\ &\quad + C\delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\ &\quad + C\delta^{-1/2} \mu_m^{-b_{|\alpha|+2}}(t) (-t)^{-1} \left(\int_0^{\underline{u}} \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \right)^{1/2}. \end{aligned} \tag{7.43}$$

This also implies an estimate for $\not{d}X_A R_i^{\alpha-1} \text{tr}\underline{\chi}$:

$$\begin{aligned}
(-t)^2 \|\not{d}X_A R_i^{\alpha-1} \text{tr}\underline{\chi}\|_{L^2(\Sigma_t^u)} &\leq C\delta \int_{-r_0}^t (-t')^{-2} \mathcal{B}_{0,\leq|\alpha|+1}(-r_0) dt' \\
&\quad + C\delta^{1/2} (-t)^{-1} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\
&\quad + C\delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\
&\quad + C\delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\
&\quad + C\delta^{-1/2} \mu_m^{-b_{|\alpha|+2}}(t) (-t)^{-1} \left(\int_0^u \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \right)^{1/2}.
\end{aligned} \tag{7.44}$$

Here we assume $b_{|\alpha|+2} \geq 4$.

7.2. Estimates for the contribution from μ . Here as in [18], to avoid the loss of derivatives when we estimate the top order spatial derivatives of μ , we need to commute \not{d} with the propagation equation of μ .

Thanks to the following commutation formulas,

$$\begin{aligned}
[L, \not{d}]\phi + \text{tr}\underline{\chi} \not{d}\phi &= -2\widehat{\underline{\chi}} \cdot \widehat{\not{D}}^2 \phi - 2\text{div}\widehat{\underline{\chi}} \cdot \not{d}\phi, \\
[T, \not{d}]\phi + c^{-1} \mu \text{tr}\theta \not{d}\phi &= -2c^{-1} \mu \widehat{\theta} \cdot \widehat{\not{D}}^2 \phi - 2\text{div}(c^{-1} \mu \widehat{\theta}) \cdot \not{d}\phi,
\end{aligned} \tag{7.45}$$

we have

$$\begin{aligned}
\underline{L}\not{d}\mu &= -\frac{1}{2} \frac{dc^2}{d\rho} \not{d}T\rho + \mu \not{d}e + e \not{d}\mu \\
&\quad + \not{d}\mu \cdot \not{d}e - \text{tr}\underline{\chi} \not{d}\mu - 2\widehat{\underline{\chi}} \cdot \widehat{\not{D}}^2 \mu - 2\text{div}\widehat{\underline{\chi}} \cdot \not{d}\mu.
\end{aligned} \tag{7.46}$$

According to (7.1),

$$\Box_g \rho = \frac{d \log(c)}{d\rho} (\mu^{-1} \underline{L}\rho L\rho + \not{d}\rho \cdot \not{d}\rho) + 2\mu^{-1} L\psi_0 \underline{L}\psi_0 + 2\not{d}\psi_0 \cdot \not{d}\psi_0.$$

Therefore, by multiplying μ , we have

$$\mu \not{d}\rho = \underline{L}(L\rho) + \frac{1}{2} \underline{L}\rho \text{tr}\chi + \frac{1}{2} L\rho \text{tr}\underline{\chi} + \frac{d \log(c)}{d\rho} (\underline{L}\rho L\rho + \mu \not{d}\rho \cdot \not{d}\rho) + 2L\psi_0 \underline{L}\psi_0 + 2\mu \not{d}\psi_0 \cdot \not{d}\psi_0.$$

We commute T and obtain

$$\begin{aligned}
\mu \not{d}T\rho &= \underline{L}(TL\rho) + \frac{1}{2} \underline{L}\rho (T \text{tr}\chi) + \frac{1}{2} L\rho (T \text{tr}\underline{\chi}) \\
&\quad + [T, \underline{L}]L\rho + \frac{1}{2} \text{tr}\chi T \underline{L}\rho + \frac{1}{2} T L\rho \text{tr}\underline{\chi} \\
&\quad + T \left(\frac{d \log(c)}{d\rho} (\underline{L}\rho L\rho + \mu \not{d}\rho \cdot \not{d}\rho) + 2L\psi_0 \underline{L}\psi_0 + 2\mu \not{d}\psi_0 \cdot \not{d}\psi_0 \right) \\
&\quad + c^{-1} \mu \text{tr}\theta \not{d}\rho + 2c^{-1} \mu \widehat{\theta} \cdot \widehat{\not{D}}^2 \rho + 2\text{div}(c^{-1} \mu \widehat{\theta}) \cdot \not{d}\rho - (T\mu) \not{d}\rho.
\end{aligned} \tag{7.47}$$

Therefore

$$\begin{aligned}
-\frac{1}{2} \frac{dc^2}{d\rho} \mu \underline{\Delta} T \rho = & \underline{L} \left(-\frac{1}{2} \frac{dc^2}{d\rho} T L \rho \right) - \frac{1}{2} \frac{dc^2}{d\rho} \left(\frac{1}{2} \underline{L} \rho (T \text{tr} \chi) + \frac{1}{2} L \rho (T \text{tr} \underline{\chi}) \right) \\
& - \frac{1}{2} \frac{dc^2}{d\rho} \left([T, \underline{L}] L \rho + \frac{1}{2} \text{tr} \chi T \underline{L} \rho + \frac{1}{2} T L \rho \text{tr} \underline{\chi} \right) \\
& - \frac{1}{2} \frac{dc^2}{d\rho} T \left(\frac{d \log(c)}{d\rho} (L \rho L \rho + \mu \not{d} \rho \cdot \not{d} \rho) + 2 L \psi_0 \underline{L} \psi_0 + 2 \mu \not{d} \psi_0 \cdot \not{d} \psi_0 \right) \\
& - \frac{1}{2} \frac{dc^2}{d\rho} \left(c^{-1} \mu \text{tr} \theta \not{D} \rho + 2 c^{-1} \mu \hat{\theta} \cdot \hat{\not{D}}^2 \rho + 2 \text{div}(c^{-1} \mu \hat{\theta}) \cdot \not{d} \rho - (T \mu) \not{D} \rho \right) \\
& + \underline{L} \left(\frac{1}{2} \frac{dc^2}{d\rho} \right) T L \rho.
\end{aligned} \tag{7.48}$$

In view of the commutator formula, we also have

$$\begin{aligned}
\mu^2 \not{D} e = & \underline{L} \left(\frac{\mu^2}{2c^2} \frac{dc^2}{d\rho} \not{D} \rho \right) + \frac{\mu^2}{c^2} \frac{dc^2}{d\rho} \left(\underline{\chi} \cdot \not{D}^2 \rho + \text{div} \hat{\chi} \cdot \not{d} \rho \right) - \underline{L} \left(\frac{\mu^2}{2c^2} \frac{dc^2}{d\rho} \right) \not{D} \rho \\
& + \mu^2 \frac{d}{d\rho} \left(\frac{1}{c^2} \frac{dc^2}{d\rho} \right) \not{d} \rho \cdot \not{d} \underline{L} \rho + \mu^2 \left(\frac{d}{d\rho} \left(\frac{1}{2c^2} \frac{dc^2}{d\rho} \right) \not{D} \rho + \frac{d^2}{d\rho^2} \left(\frac{1}{2c^2} \frac{dc^2}{d\rho} \right) |\not{d} \rho|^2 \right) \underline{L} \rho.
\end{aligned} \tag{7.49}$$

With the same notation as in [18], we define:

$$\check{f}' := -\frac{1}{2} \frac{dc^2}{d\rho} T L \rho + \frac{\mu^2}{2c^2} \frac{dc^2}{d\rho} \not{D} \rho, \quad F' := \mu \not{D} \mu - \check{f}'. \tag{7.50}$$

and we have the following propagation equation for F' :

$$\underline{L} F' + (\text{tr} \underline{\chi} - 2\mu^{-1} \underline{L} \mu) F' = - \left(\frac{1}{2} \text{tr} \underline{\chi} - 2\mu^{-1} \underline{L} \mu \right) \check{f}' - 2\mu \hat{\chi} \cdot \hat{\not{D}}^2 \mu + \check{g}'. \tag{7.51}$$

where

$$\check{g}' = \left(-\not{d} \mu + \frac{\mu}{c^2} \frac{dc^2}{d\rho} \not{d} \rho \right) \cdot (\mu \not{d} \text{tr} \underline{\chi}) + \Psi_{\geq -2,3}^{\leq 2} + \mathcal{O}_{0,1}^{\leq 1} \Psi_{\geq -2,2}^{\leq 2} + \Psi_{\geq 0,4}^{\leq 2}. \tag{7.52}$$

Similarly as in [18], we have used the structure equation (2.32) to cancel the contribution from the term $\frac{1}{2} \underline{L} \rho (T \text{tr} \chi) + \frac{1}{2} L \rho (T \text{tr} \underline{\chi})$ in (7.47) and the term $(\underline{L} \mu) \not{D} \mu$ when we write $\underline{L}(\mu \not{D} \mu) = \mu \underline{L}(\not{D} \mu) + \underline{L}(\mu) \not{D} \mu$. We also remark that the L^2 norm of all derivatives on $\text{div} \hat{\chi}$ has been estimated from previous subsection. In such a sense, it can also be considered as a $\Psi_{2,4}^{\leq 2}$ term and we use (2.31) to replace $\text{div} \hat{\chi}$ by $\not{d} \text{tr} \underline{\chi} + \dots$. The term $\Psi_{\geq -2,3}^{\leq 2}$ comes from the contribution of $L \psi_0 \underline{L} \psi_0$ and $\mathcal{O}_{0,1}^{\leq 1} \Psi_{\geq -2,2}^{\leq 2}$ comes from $-\frac{1}{4} \frac{dc^2}{d\rho} T L \rho \text{tr} \underline{\chi}$ in (7.48). Since we already applied T to $L \psi_0 \underline{L} \psi_0$ once in (7.48), instead of using flux $\underline{F}(t, \underline{u})$ as we did in the last subsection, we only need to use the energy $E(t, \underline{u})$ to control the contribution of this term. For the higher order derivatives $F'_{\alpha,l} = \mu X_A R_i^{\alpha'-1} T^l \not{D} \mu - X_A R_i^{\alpha'-1} T^l \check{f}'$ with $|\alpha'| + l = |\alpha|$, we have:

$$\underline{L} F'_{\alpha,l} + (\text{tr} \underline{\chi} - 2\mu^{-1} \underline{L} \mu) F'_{\alpha,l} = - \left(\frac{1}{2} \text{tr} \underline{\chi} - 2\mu^{-1} \underline{L} \mu \right) X_A R_i^{\alpha'-1} T^l \check{f}' - 2\mu \hat{\chi} \cdot \not{L}_{X_A} \not{L}_{R_i}^{\alpha'-1} \not{L}_T^l \hat{\not{D}}^2 \mu + \check{g}'_{\alpha',l}. \tag{7.53}$$

where $\check{g}'_{\alpha',l}$ is given by

$$\begin{aligned}
\check{g}'_{\alpha',l} = & \left(-\not{d} \mu + \frac{\mu}{c^2} \frac{dc^2}{d\rho} \not{d} \rho \right) \cdot \mu \not{d} \left(X_A R_i^{\alpha'-1} T^l \text{tr} \underline{\chi} \right) + l \cdot \Lambda F'_{\alpha,l-1} + {}^{(R_i)} \underline{L} F'_{\alpha-1,l} \\
& + \mathcal{O}_{-k,\geq 1}^{\leq |\beta|+1} \mathcal{O}_{\geq 2-2l+k,\geq 4}^{\leq |\alpha|-|\beta|+1} + \Psi_{\geq -2l-2,4}^{\leq |\alpha|+2} + \mathcal{O}_{-k,1}^{\leq |\beta|+1} \Psi_{\geq -2l-2+k,3}^{\leq |\alpha|+2-|\beta|} + \Psi_{\geq -2l,4}^{\leq |\alpha|+2}.
\end{aligned}$$

Note that $F'_{\alpha,l}$ is a scalar function instead of a 1-form, so the inequality for $|F'_{\alpha,l}(t, \underline{u})|$ is slightly different from that of $|F_\alpha|$ as in the last subsection. We have:

$$\underline{L}|F'_{\alpha,l}| + (\text{tr}\underline{\chi} - 2\mu^{-1}\underline{L}\mu) |F'_{\alpha,l}| \leq \left(-\frac{1}{2}\text{tr}\underline{\chi} + 2\mu^{-1}|\underline{L}\mu|\right) |X_A R_i^{\alpha'-1} T^l \check{f}'| + \tilde{g}'_{\alpha',l}, \quad (7.54)$$

with

$$\begin{aligned} \tilde{g}'_{\alpha',l} &:= 2 \left| \mu \hat{\chi} \cdot \not\!L_{X_A} \not\!L_{R_i}^{\alpha'-1} \not\!L_T^l \hat{\mathcal{D}}^2 \mu \right| + |\check{g}'_{\alpha,l}| \\ \partial_t \left((-t + \underline{u})^2 |F'_{\alpha,l}| \right) &+ (\text{tr}\underline{\chi} - 2\mu^{-1}\underline{L}\mu) (-t + \underline{u})^2 |F'_{\alpha,l}| \\ &\leq (-t + \underline{u})^2 (2\mu^{-1}|\partial_t \mu| - \text{tr}\underline{\chi}) |X_A R_i^{\alpha'-1} T^l \check{f}'| + (-t + \underline{u})^2 \tilde{g}'_{\alpha',l}, \end{aligned} \quad (7.55)$$

Integrating (7.55) and taking the L^2 norm on $[0, \underline{u}] \times \mathbb{S}^2$, we obtain:

$$\begin{aligned} (-t)^2 \delta^{l+1} \|F'_{\alpha,l}(t)\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} &\lesssim r_0^2 \delta^{l+1} \|F'_{\alpha,l}(-r_0)\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} \\ &+ \delta^{l+1} \int_{-r_0}^t (-t' + \underline{u})^2 (2\mu^{-1}|\partial_t \mu| - \text{tr}\underline{\chi}) \|X_A R_i^{\alpha'-1} T^l \check{f}'(t')\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} dt' \\ &+ \delta^{l+1} \int_{-r_0}^t (-t' + \underline{u})^2 \|\tilde{g}'_{\alpha',l}(t')\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} dt' := I' + II' + III'. \end{aligned} \quad (7.56)$$

Since I' depends only on initial data, here we will focus on II' and III' . In view of the definition of \check{f}' (7.50),

$$\delta^{l+1} \|X_A R_i^{\alpha'-1} T^l \check{f}'\|_{L^2(\Sigma_t^{\underline{u}})} \lesssim \delta^{1/2} (-t)^{-2} \sqrt{E_{\leq |\alpha|+2}(t, \underline{u})} + \delta^{3/2} (-t)^{-3} \sqrt{E_{\leq |\alpha|+2}(t, \underline{u})}.$$

This together with Lemma 7.1 and the pointwise estimates for $\text{tr}\underline{\chi}$ as well as $\underline{L}\mu$, we have:

$$\begin{aligned} II' &\lesssim \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) (-t)^{-1} \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} + \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt' \\ &+ \delta^{3/2} \int_{-r_0}^t (-t')^{-3} \sqrt{E_{\leq |\alpha|+2}(t', \underline{u})} dt'. \end{aligned} \quad (7.57)$$

Next we move to the L^2 norm of $\check{g}'_{\alpha',l}$. First, in view of the pointwise estimates $|\Lambda| \lesssim (-t)^{-2}$, $|(R_i) \underline{Z}| \lesssim \delta (-t)^{-2}$, the two terms $l \cdot \Lambda F'_{\alpha,l-1}$, $(R_i) \underline{Z} F'_{\alpha-1,l}$ are controlled by Gronwall. If $l = 0$, in view of (7.44) and the pointwise estimates for $\not\!L\mu$ and $\not\!L\rho$, the contribution of the first term in the expression of $\check{g}'_{\alpha',l}$ to III' can be bounded as

$$\begin{aligned} &\delta^{3/2} \mu_m^{-b_{|\alpha|+2}}(t) (-t)^{-1} \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \\ &+ \delta^{3/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt' \\ &+ \delta^{3/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt' \\ &+ \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) (-t)^{-1} \left(\int_0^{\underline{u}} \tilde{F}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}' \right)^{1/2}. \end{aligned} \quad (7.58)$$

If $l \geq 1$, then in view of Proposition 6.4, the pointwise estimates for $\not{d}\mu$, $\not{d}\rho$ and the structure equation (2.32), the contribution of this term is bounded as:

$$\begin{aligned} & \delta^2 \int_{-r_0}^t (-t')^{-3} \left(\mathcal{B}_{l-1, \leq |\alpha|+2}(t') + \sqrt{E_{\leq |\alpha|+2}(t', \underline{u})} + \mu_m^{-1/2}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} \right) dt' \\ & \lesssim \delta^2 \int_{-r_0}^t (-t')^{-3} \left(\mathcal{B}_{l-1, \leq |\alpha|+2}(-r_0) + \sqrt{E_{\leq |\alpha|+2}(t', \underline{u})} + \mu_m^{-1/2}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} \right) dt'. \end{aligned} \quad (7.59)$$

Now we move to the second line of the expression of $\check{g}'_{\alpha', l}$. Again, the term $\mathcal{O}_{-k, \geq 1}^{\leq |\beta|+1} \mathcal{O}_{\geq 2-2l+k, \geq 4}^{\leq |\alpha|-|\beta|+1}$ can be absorbed by $\mathcal{O}_{-k, 1}^{\leq |\beta|+1} \Psi_{\geq -2l-2+k, 3}^{\leq |\alpha|+2-|\beta|}$. So we only need to consider the last three terms in this expression. In view of Proposition 6.3 and Proposition 6.4, the contributions of these terms to III' are bounded as follows:

$$\delta^{l+1} \int_{-r_0}^t (-t') \|\Psi_{\geq -2l-2, 4}^{\leq |\alpha|+2}\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \lesssim \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \sqrt{E_{\leq |\alpha|+2}(t', \underline{u})} dt' \quad (7.60)$$

$$\begin{aligned} & \delta^{l+1} \int_{-r_0}^t (-t') \|\mathcal{O}_{-k, 1}^{\leq |\beta|+1} \Psi_{\geq -2l-2+k, 3}^{\leq |\alpha|+2-|\beta|}\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \\ & \lesssim \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \sqrt{E_{\leq |\alpha|+2}(t', \underline{u})} dt' + \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-1/2}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt'. \end{aligned} \quad (7.61)$$

and

$$\begin{aligned} & \delta^{l+1} \int_{-r_0}^t (-t') \|\Psi_{\geq -2l, 4}^{\leq |\alpha|+2}\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \lesssim \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-1/2}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt' \\ & \quad + \delta^{3/2} \int_{-r_0}^t (-t')^{-2} \sqrt{E_{\leq |\alpha|+2}(t', \underline{u})} dt'. \end{aligned} \quad (7.62)$$

Now combining (7.56) and (7.57)-(7.62), we obtain the following estimates:

$$\begin{aligned} & \delta^{l+1} (-t) \|F'_{\alpha, l}(t)\|_{L^2(\Sigma_t^{\underline{u}})} \lesssim \delta^{l+1} r_0 \|F'_{\alpha, l}(-r_0)\|_{L^2(\Sigma_{-r_0}^{\underline{u}})} \\ & \quad + \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) (-t)^{-1} \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \\ & \quad + \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt' \\ & \quad + \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt' \\ & \quad + \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) (-t)^{-1} \left(\int_0^{\underline{u}} \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}' \right)^{1/2}. \end{aligned} \quad (7.63)$$

These in terms imply the estimates for $\delta^{l+1}\|\mu R^{\alpha'} T^l \Delta \mu\|_{L^2(\Sigma_t^{\underline{u}})}$:

$$\begin{aligned}
\delta^{l+1}\|\mu R^{\alpha'} T^l \Delta \mu\|_{L^2(\Sigma_t^{\underline{u}})} &\lesssim \delta^{l+1} r_0 \|F'_{\alpha,t}(-r_0)\|_{L^2(\Sigma_{-r_0}^{\underline{u}})} \\
&\quad + \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) (-t)^{-1} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\
&\quad + \delta^{3/2} (-t)^{-2} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\
&\quad + \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\
&\quad + \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\
&\quad + \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) (-t)^{-1} \left(\int_0^{\underline{u}} \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \right)^{1/2}.
\end{aligned} \tag{7.64}$$

8. COMMUTATOR ESTIMATES

In this section, we shall estimate the error spacetime integrals contributed by commutators.

Let ψ be a solution of the inhomogeneous wave equation $\square_{\tilde{g}} \psi = \rho$ and Z be a vector field, one can commute Z with the equation to derive

$$\square_{\tilde{g}}(Z\psi) = Z\rho + \frac{1}{2} \text{tr}_{\tilde{g}}^{(Z)} \tilde{\pi} \cdot \rho + c^2 \text{div}_g^{(Z)} J \tag{8.1}$$

where the vector field $^{(Z)}J$ is defined by

$$^{(Z)}J^\mu = \left(^{(Z)}\tilde{\pi}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \text{tr}_g^{(Z)} \tilde{\pi} \right) \partial_\nu \psi.$$

We remark that the raising indices for $^{(Z)}\tilde{\pi}^{\mu\nu}$ are with respect to the optic metric g .

In applications, we use the above formulas for homogeneous wave equations $\square_{\tilde{g}} \psi = 0$ and commute some commutation vector fields Z_i 's several times. Therefore, we need the following recursion formulas:

$$\begin{aligned}
\square_{\tilde{g}} \psi_n &= \rho_n, \quad \psi_n = Z\psi_{n-1}, \quad \rho_1 = 0, \\
\rho_n &= Z\rho_{n-1} + \frac{1}{2} \text{tr}_{\tilde{g}}^{(Z)} \tilde{\pi} \cdot \rho_{n-1} + c^2 \text{div}_g^{(Z)} J_{n-1}, \\
^{(Z)}J_{n-1}^\mu &= \left(^{(Z)}\tilde{\pi}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \text{tr}_g^{(Z)} \tilde{\pi} \right) \partial_\nu \psi_{n-1}.
\end{aligned} \tag{8.2}$$

Remark 8.1. When we derive energy estimates for $\square_{\tilde{g}} \psi_n = \rho_n$, due to the volume form of the conformal optic metric \tilde{g} , the integrands $\tilde{\rho}_n$ appearing in the error terms is slightly different from ρ_n . The rescaled source terms $\tilde{\rho}_n$ are defined as follows:

$$\begin{aligned}
\tilde{\rho}_n &= \frac{1}{c^2} \mu \rho_n = Z\tilde{\rho}_{n-1} + ^{(Z)}\delta \cdot \tilde{\rho}_{n-1} + ^{(Z)}\sigma_{n-1}, \\
\tilde{\rho}_1 &= 0, \quad ^{(Z)}\sigma_{n-1} = \mu \cdot \text{div}_g^{(Z)} J_{n-1}, \quad ^{(Z)}\delta = \frac{1}{2} \text{tr}_g^{(Z)} \tilde{\pi} - \mu^{-1} Z\mu + 2Z(\log(c)).
\end{aligned} \tag{8.3}$$

Then the error spacetime integrals corresponding to $K_0 = L + \underline{L}$ and $K_1 = \frac{2(-t)}{\widetilde{\text{tr}}\underline{\chi}}\underline{L}$ containing ρ_n are as follows:

$$\begin{aligned} & - \int_{W_{\underline{u}}^t} \frac{1}{c^2} \rho_n K_0 \psi_n d\mu_g = - \int_{W_{\underline{u}}^t} \tilde{\rho}_n (L\psi_n + \underline{L}\psi_n) dt d\underline{u} d\mu_g \\ & - \int_{W_{\underline{u}}^t} \frac{1}{c^2} \rho_n (K_1 \psi_n - t\psi_n) d\mu_g = - \int_{W_{\underline{u}}^t} \tilde{\rho}_n (K_1 \psi_n - t\psi_n) dt d\underline{u} d\mu_g \end{aligned}$$

We first consider the contribution of ${}^{(Z)}\sigma_{n-1}$ in $\tilde{\rho}_n$. We write ${}^{(Z)}\sigma_{n-1}$ in null frame $(\underline{L}, L, \frac{\partial}{\partial\theta^A})$:

$$\begin{aligned} {}^{(Z)}\sigma_{n-1} &= -\frac{1}{2}L({}^{(Z)}J_{n-1,\underline{L}}) - \frac{1}{2}\underline{L}({}^{(Z)}J_{n-1,L}) + \text{div}(\mu^{(Z)}\not{J}_{n-1}) \\ &\quad - \frac{1}{2}\underline{L}(c^{-2}\mu)^{(Z)}J_{n-1,\underline{L}} - \frac{1}{2}\text{tr}\chi^{(Z)}J_{n-1,\underline{L}} - \frac{1}{2}\text{tr}\underline{\chi}^{(Z)}J_{n-1,L}, \end{aligned}$$

Then with the following expressions for the components of ${}^{(Z)}J_{n-1}$ in the null frame:

$$\begin{aligned} {}^{(Z)}J_{n-1,\underline{L}} &= -\frac{1}{2}\text{tr}^{(Z)}\tilde{\not{J}}(\underline{L}\psi_{n-1}) + {}^{(Z)}\tilde{\underline{Z}} \cdot \not{d}\psi_{n-1} \\ {}^{(Z)}J_{n-1,L} &= -\frac{1}{2}\text{tr}^{(Z)}\tilde{\not{J}}(L\psi_{n-1}) + {}^{(Z)}\tilde{Z} \cdot \not{d}\psi_{n-1} - \frac{1}{2\mu}{}^{(Z)}\tilde{\pi}_{LL}(\underline{L}\psi_{n-1}) \\ \mu^{(Z)}\not{J}_{n-1}^A &= -\frac{1}{2}{}^{(Z)}\tilde{Z}^A(\underline{L}\psi_{n-1}) - \frac{1}{2}{}^{(Z)}\tilde{\underline{Z}}^A(L\psi_{n-1}) \\ &\quad + \frac{1}{2}({}^{(Z)}\tilde{\pi}_{L\underline{L}} - \mu\text{tr}^{(Z)}\tilde{\not{J}})\not{d}^A\psi_{n-1} + \mu^{(Y)}\tilde{\not{J}}_B^A\not{d}^B\psi_{n-1} \end{aligned}$$

Based on the above expressions, we decompose:

$${}^{(Z)}\sigma_{n-1} = {}^{(Z)}\sigma_{1,n-1} + {}^{(Z)}\sigma_{2,n-1} + {}^{(Z)}\sigma_{3,n-1}$$

where ${}^{(Z)}\sigma_{1,n-1}$ contains the products of components of ${}^{(Z)}\tilde{\pi}$ with the 2nd derivatives of ψ_{n-1} , ${}^{(Z)}\sigma_{2,n-1}$ contains the products of the 1st derivatives of ${}^{(Z)}\tilde{\pi}$ with the 1st derivatives of ψ_{n-1} , and ${}^{(Z)}\sigma_{3,n-1}$ contains the other lower order terms. More specifically, we have:

$$\begin{aligned} {}^{(Z)}\sigma_{1,n-1} &= \frac{1}{2}\text{tr}^{(Z)}\tilde{\not{J}}(\underline{L}L\psi_{n-1}) + \frac{1}{2}\text{tr}_{\not{g}}\tilde{\chi}L\psi_{n-1} \\ &\quad + \frac{1}{4}(\mu^{-1}{}^{(Z)}\tilde{\pi}_{LL})\underline{L}^2\psi_{n-1} \\ &\quad - {}^{(Z)}\tilde{Z} \cdot \not{d}\underline{L}\psi_{n-1} - {}^{(Z)}\tilde{\underline{Z}} \cdot \not{d}L\psi_{n-1} \\ &\quad + \frac{1}{2}{}^{(Z)}\tilde{\pi}_{L\underline{L}}\not{d}\psi_{n-1} + \mu^{(Z)}\hat{\pi} \cdot \not{D}^2\psi_{n-1} \end{aligned} \tag{8.4}$$

$$\begin{aligned} {}^{(Z)}\sigma_{2,n-1} &= \frac{1}{4}\underline{L}(\text{tr}^{(Z)}\tilde{\not{J}})L\psi_{n-1} + \frac{1}{4}L(\text{tr}^{(Z)}\tilde{\not{J}})\underline{L}\psi_{n-1} \\ &\quad + \frac{1}{4}\underline{L}(\mu^{-1}{}^{(Z)}\tilde{\pi}_{LL})\underline{L}\psi_{n-1} \\ &\quad - \frac{1}{2}\not{L}\underline{L}{}^{(Z)}\tilde{Z} \cdot \not{d}\psi_{n-1} - \frac{1}{2}\not{L}{}^{(Z)}\tilde{\underline{Z}} \cdot \not{d}\psi_{n-1} \\ &\quad - \frac{1}{2}\text{div}^{(Z)}\tilde{Z}\underline{L}\psi_{n-1} - \frac{1}{2}\text{div}^{(Z)}\tilde{\underline{Z}}L\psi_{n-1} \\ &\quad + \frac{1}{2}\not{d}^{(Z)}\tilde{\pi}_{L\underline{L}}\not{d}\psi_{n-1} + \text{div}(\mu^{(Z)}\hat{\pi}) \cdot \not{d}\psi_{n-1} \end{aligned} \tag{8.5}$$

and

$$\begin{aligned}
^{(Z)}\sigma_{3,n-1} = & \left(\frac{1}{4} \text{tr} \chi \text{tr}^{(Z)} \tilde{\pi} + \frac{1}{4} \text{tr} \underline{\chi} (\mu^{-1})^{(Z)} \tilde{\pi}_{LL} \right) \\
& + \frac{1}{2} {}^{(Z)} \tilde{\underline{Z}} \cdot \underline{\mathcal{A}}(c^{-2} \mu) \underline{L} \psi_{n-1} - \frac{1}{4} (\underline{L} \log(c^{-1})) \text{tr}^{(Z)} \tilde{\pi} \underline{L} \psi_{n-1} \\
& - \left(\left(\frac{1}{2} \text{tr} \chi + \underline{L}(c^{-2} \mu) \right) {}^{(Z)} \tilde{\underline{Z}} + \frac{1}{2} \text{tr} \underline{\chi} {}^{(Z)} \tilde{\underline{Z}} \right) \underline{\mathcal{A}} \psi_{n-1}
\end{aligned} \tag{8.6}$$

With these expressions for $^{(Z)}\sigma_{n-1}$, we are able to investigate the structure of $\tilde{\rho}_n$. Basically, we want to use the recursion formulas in (8.3) to obtain a relatively explicit expression for $\tilde{\rho}_n$.

On the other hand, for the energy estimates, we consider the following possible ψ_n :

$$\psi_n = R_i^{\alpha+1} \psi, \quad \psi_n = R_i^{\alpha'} T^{l+1} \psi, \quad \psi_n = Q R_i^{\alpha'} T^l \psi$$

Here ψ_n is the n th order variation and $n = |\alpha| + 1 = |\alpha'| + l + 1$ and ψ is any first order variation. The reason that we can always first apply T , then R_i , and finally a possible Q is that the commutators $[R_i, T]$, $[R_i, Q]$, and $[T, Q]$ are one order lower than $R_i T, T R_i$; $Q R_i, R_i Q$; and $Q T, T Q$ respectively. Moreover, all these commutators are tangent to $S_{t,\underline{u}}$. Since we let Q be the last possible commutator, there will be no Q 's in ψ_{n-1} in the second term on the right hand side of (8.4). Therefore we only need to commute Q once.

Now suppose that we consider the variations of order $n = |\alpha| + 2$ in the following form:

$$\psi_{|\alpha|+2} := Z_{|\alpha|+1} \dots Z_1 \psi.$$

We have the inhomogeneous wave equation:

$$\square_{\tilde{g}} \psi_{|\alpha|+2} = \rho_{|\alpha|+2}$$

As we pointed out in Remark 8.1, we define:

$$\tilde{\rho}_{|\alpha|+2} = \frac{\mu}{c^2} \rho_{|\alpha|+2}$$

Then by a induction argument, the corresponding inhomogeneous term $\tilde{\rho}_{|\alpha|+2}$ is given by:

$$\tilde{\rho}_{|\alpha|+2} = \sum_{k=0}^{|\alpha|} (Z_{|\alpha|+1} + {}^{(Z_{|\alpha|+1})} \delta) \dots (Z_{|\alpha|-k+2} + {}^{(Z_{|\alpha|-k+2})} \delta) {}^{(Z_{|\alpha|-k+1})} \sigma_{|\alpha|-1+k} \tag{8.7}$$

8.1. Error Estimates for the lower order terms. Consider an arbitrary term in the sum (8.7). There is a total of k derivatives with respect to the commutators acting on $^{(Z)}\sigma_{|\alpha|-1+k}$. In view of the fact that $^{(Z)}\sigma_{|\alpha|-1+k}$ has the structure described in (8.4), (8.5) and (8.6), in considering the partial contribution of each term in $^{(Z)}\sigma_{1,|\alpha|-1+k}$, if the factor which is a component of $^{(Z)}\tilde{\pi}$ receives more than $\lceil \frac{|\alpha|+1}{2} \rceil$ derivatives with respect to the commutators, then the factor which is a 2nd order derivative of $\psi_{|\alpha|+1-k}$ receives at most $k - \lceil \frac{|\alpha|+1}{2} \rceil - 1$ order derivatives of commutators, thus corresponds to a derivative of the ψ of order at most: $k - \lceil \frac{|\alpha|+1}{2} \rceil + 1 + |\alpha| - k = \lceil \frac{|\alpha|}{2} \rceil + 1$, therefore this factor is bounded in $L^\infty(\Sigma_t^u)$ by the bootstrap assumption. Also, in considering the partial contribution of each term in $^{(Z)}\sigma_{2,|\alpha|+1-k}$, if the factor which is a 1st derivative of $^{(Z)}\tilde{\pi}$ receives more than $\lceil \frac{|\alpha|+1}{2} \rceil - 1$ derivatives with respect to the commutators, then the factor which is a 1st derivative of $\psi_{|\alpha|+1-k}$ receives at most $k - \lceil \frac{|\alpha|+1}{2} \rceil$ derivatives with respect to the commutators, thus corresponds to a derivative of the ψ_α of order at most $k - \lceil \frac{|\alpha|+1}{2} \rceil + 1 + |\alpha| - k = \lceil \frac{|\alpha|}{2} \rceil + 1$, therefore this factor is again bounded in $L^\infty(\Sigma_t^u)$ by the bootstrap assumption. Similar considerations apply to $^{(Z)}\sigma_{3,|\alpha|+1-k}$. We

conclude that for all the terms in the sum in (8.7) of which one factor is a derivative of the $^{(Z)}\tilde{\pi}$ of order more than $\lfloor \frac{|\alpha|+1}{2} \rfloor$, the other factor is then a derivative of the ψ_α of order at most $\lfloor \frac{|\alpha|}{2} \rfloor + 1$ and is thus bounded in $L^\infty(\Sigma_t^u)$ by the bootstrap assumption. Of these terms we shall estimate the contribution of those containing the top order spatial derivatives of the optical entities in the next subsection. Before we give the estimates for the contribution of the lower order optical terms to the spacetime integrals:

$$-\delta^{2k} \int_{W_{\underline{u}}^t} \tilde{\rho}_{\leq|\alpha|+2} (L\psi_{\leq|\alpha|+2} + \underline{L}\psi_{\leq|\alpha|+2}) dt d\underline{u} d\mu_{\mathcal{G}}, \quad -\delta^{2k} \int_{W_{\underline{u}}^t} \tilde{\rho}_{\leq|\alpha|+2} \left(\frac{2(-t')}{\tilde{\text{tr}}\underline{\chi}} \underline{L}\psi_{\leq|\alpha|+2} - t'\psi_{\leq|\alpha|+2} \right) dt d\underline{u} d\mu_{\mathcal{G}}, \quad (8.8)$$

we investigate the behavior of these integrals with respect to δ . Here k is the number of T s in string of commutators. For the multiplier $K_1 = \frac{2(-t')}{\tilde{\text{tr}}\underline{\chi}} \underline{L}$, the associated energy inequality is

$$\underline{E}_{\leq|\alpha|+2}(t, \underline{u}) + \underline{F}_{\leq|\alpha|+2}(t, \underline{u}) + K_{\leq|\alpha|+2}(t, \underline{u}) \lesssim \underline{E}_{\leq|\alpha|+2}(-r_0, \underline{u}) + \int_{W_{\underline{u}}^t} \tilde{Q}_{1, \leq|\alpha|+2}. \quad (8.9)$$

The quantities $K_{\leq|\alpha|+2}(t, \underline{u})$ are defined similar as $K(t, \underline{u})$:

$$K_{\leq|\alpha|+2}(t, \underline{u}) := \sum_{|\alpha'| \leq |\alpha|+1} \delta^{l'} K(t, \underline{u}) [Z^{\alpha'} \psi],$$

Again, l' is the number of T 's in $Z^{\alpha'}$.

In $\tilde{Q}_{1, \leq|\alpha|+2}$, there are contributions from the deformation tensors of two multipliers, which has been treated in Section 5. There are also contributions from the deformation tensors of commutators, which are given by (8.7). Now we investigate the terms which are not top order optical terms, namely, the terms containing $\underline{\chi}$ and μ of order less than $|\alpha|+2$. In view of the discussion in Section 5, the left hand side of (8.9) is of order δ , so we expect these lower order terms in the second integral of (8.8) is at least of order δ . In fact the integration on $W_{\underline{u}}^t$ gives us a δ and the contribution from the variation $\delta^k \left(\frac{2(-t')}{\tilde{\text{tr}}\underline{\chi}} \underline{L}\psi_{\leq|\alpha|+2} - t'\psi_{\leq|\alpha|+2} \right)$ is of order $\delta^{1/2}$. For the behavior of $\delta^k \sigma$, we take a look at σ_1 as an example. Let k' be the number of T s applied to $(\underline{L}L\psi_l + \frac{1}{2}\tilde{\text{tr}}\underline{\chi}L\psi_l)$. (4.1) implies

$$\delta^{k'} \left(\underline{L}L\psi_l + \frac{1}{2}\tilde{\text{tr}}\underline{\chi}L\psi_l \right) \sim \delta^{1/2} + \delta^{k'} \rho_l.$$

Since $\rho_1 = 0$, an induction argument implies that

$$\delta^{k'} \left(\underline{L}L\psi_l + \frac{1}{2}\tilde{\text{tr}}\underline{\chi}L\psi_l \right) \sim \delta^{1/2}.$$

Then in view of (3.25), (3.26) and (3.30), the first term in σ_1 behaves like $\delta^{1/2}$. Following the same procedure, one sees straightforwardly that all the other terms in σ_1, σ_2 and σ_3 behave like $\delta^{1/2}$ (one keeps in mind that if $Z = T$, then we multiplier a δ with the corresponding deformation tensor.) except the term $\underline{L} \left(\text{tr}^{(Z)} \tilde{\pi} \right) L\psi_l$. For this term we use the argument deriving (3.29) and Proposition 6.1 to see actually we have:

$$\|\underline{L}(\text{tr}^{(Q)} \tilde{\pi})\|_{L^\infty(\Sigma_t^u)} \lesssim \delta.$$

This completes the discussions for σ associated to K_1 .

The same argument applies to the energy inequality associated to K_0 :

$$E_{\leq|\alpha|+2}(t, \underline{u}) + F_{\leq|\alpha|+2}(t, \underline{u}) \lesssim E_{\leq|\alpha|+2}(-r_0, \underline{u}) + \int_{W_{\underline{u}}^t} \tilde{Q}_{0, \leq|\alpha|+2}. \quad (8.10)$$

and we conclude that the lower order optical terms in the error spacetime integrals have one more power in δ than the energies on the left hand side.

Now we summarize the spacetime error estimates for the terms which come from the L^2 norms of the lower order optical quantities. In view of the proof for Proposition 6.3 and 6.4 and the bootstrap assumption on L^∞ norms of variations, the contribution to the spacetime error integral from the L^2 norms of the lower order optical terms is bounded as:

$$\begin{aligned} & \int_{-r_0}^t (-t')^{-2} \left(\sum_{|\alpha'| \leq |\alpha|+1} \delta^{1/2+2l} \|\not{d}Z_i^{\alpha'} \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} + \delta^{1/2} \sqrt{E_{\leq|\alpha|+2}(t', \underline{u})} \right) dt' \\ & \cdot \delta^{1/2} \int_{-r_0}^t (-t')^{-1} \left\| \left(\underline{L}\psi_{|\alpha|+2} + \frac{1}{2} \tilde{\text{tr}} \chi \psi_{|\alpha|+2} \right) \right\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \\ & \lesssim \int_{-r_0}^t (-t')^{-2} \left(\sum_{|\alpha'| \leq |\alpha|+1} \delta^{2+2l} \|\not{d}Z_i^{\alpha'} \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})}^2 + \delta^2 E_{\leq|\alpha|+2}(t', \underline{u}) \right) dt' \\ & + \int_0^{\underline{u}} \underline{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \\ & \lesssim \delta^2 \int_{-r_0}^t (-t')^{-2} E_{\leq|\alpha|+2}(t', \underline{u}) dt' + \int_0^{\underline{u}} \underline{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' + \delta^2 K_{\leq|\alpha|+2}(t, \underline{u}) \quad \text{for } K_1. \end{aligned} \quad (8.11)$$

and

$$\begin{aligned} & \int_{-r_0}^t (-t')^{-2} \left(\sum_{|\alpha'| \leq |\alpha|+1} \delta^{1/2+2l} \|\not{d}Z_i^{\alpha'} \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} + \delta^{1/2} \sqrt{E_{\leq|\alpha|+2}(t', \underline{u})} \right) dt' \\ & \cdot \delta^{1/2} \int_{-r_0}^t (-t')^{-3/2} \left\| \underline{L}\psi_{|\alpha|+2} + L\psi_{|\alpha|+2} \right\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \\ & \lesssim \int_{-r_0}^t (-t')^{-2} \left(\sum_{|\alpha'| \leq |\alpha|+1} \delta^{1+2l} \|\not{d}Z_i^{\alpha'} \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})}^2 + \delta E_{\leq|\alpha|+2}(t', \underline{u}) \right) dt' \\ & + \delta \int_{-r_0}^t (-t')^{-3/2} E_{\leq|\alpha|+2}(t', \underline{u}) dt' \\ & \lesssim \delta \int_{-r_0}^t (-t')^{-3/2} E_{\leq|\alpha|+2}(t', \underline{u}) dt' + \delta K_{\leq|\alpha|+2}(t, \underline{u}), \quad \text{for } K_0. \end{aligned} \quad (8.12)$$

Next we consider the case in which the deformation tensors receive less derivatives such that they can be bounded in L^∞ . More specifically, we consider the terms in the sum (8.7) in which there are at most $\lceil \frac{|\alpha|+1}{2} \rceil$ derivatives hitting the deformation tensor $^{(Z)}\tilde{\not{d}}\not{\chi}$, thus the spatial derivatives on $\underline{\chi}$ is at most $\lceil \frac{|\alpha|+1}{2} \rceil$ and the spatial derivatives on μ is at most $\lceil \frac{|\alpha|+1}{2} \rceil + 1$, which are bounded in $L^\infty(\Sigma_t^{\underline{u}})$.

Let us start with the contribution associated to K_1 . In view of (3.27), (3.29) and (3.40), the contribution from the first line in (8.4) is bounded by

$$\int_0^{\underline{u}} \underline{F}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}'. \quad (8.13)$$

In view of (3.27), (3.28) and (3.40), the contribution from the second line in (8.4) is bounded by (up to a constant)

$$\int_0^{\underline{u}} \underline{F}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}' + \delta \int_{-r_0}^t (-t')^{-2} \underline{E}_{\leq |\alpha|+2}(t', \underline{u}) dt'. \quad (8.14)$$

The contribution from the first term in the third line of (8.4) enjoys the same estimate as (8.14), while the contribution of the second term is bounded by (up to a constant)

$$\delta \int_{-r_0}^t (-t')^{-2} \underline{E}_{\leq |\alpha|+2}(t', \underline{u}) dt' + \int_0^{\underline{u}} \underline{F}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}' \quad (8.15)$$

in view of (3.27) and (3.40). The contribution from the last line in (8.4) is bounded by (up to a constant)

$$\delta^{1/2} K_{\leq |\alpha|+2}(t, \underline{u}) + \delta^{-1/2} \int_0^{\underline{u}} \underline{F}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}' + \int_{-r_0}^t (-t')^{-2} \underline{E}_{\leq |\alpha|+2}(t', \underline{u}) dt'. \quad (8.16)$$

Here besides (3.27), (3.28) and (3.40), we also used the inequality $ab \leq \frac{1}{2}(\delta^{1/2}a^2 + \delta^{-1/2}b^2)$.

The contribution of the third, fourth and fifth lines in (8.5) is bounded in the similar way as the third and fourth lines in (8.4). The second term in the first line of (8.5) can be bounded in the similar way as the first line of (8.4). The contributions from the first term in the first line and the second line of (8.5) are bounded by (up to a constant)

$$\int_0^{\underline{u}} \underline{F}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}' + \delta \int_{-r_0}^t (-t')^{-2} \underline{E}_{\leq |\alpha|+2}(t', \underline{u}) dt'. \quad (8.17)$$

Since the contributions from (8.6) are of lower order compared to the contributions of (8.4) and (8.5), we omit the details. Therefore the contributions associated to K_1 are bounded by (up to a constant)

$$\begin{aligned} & \delta^{1/2} K_{\leq |\alpha|+2}(t, \underline{u}) + \delta^{-1/2} \int_0^{\underline{u}} \underline{F}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}' \\ & + \int_{-r_0}^t (-t')^{-2} \underline{E}_{\leq |\alpha|+2}(t', \underline{u}) dt' + \delta \int_{-r_0}^t (-t')^{-2} \underline{E}_{\leq |\alpha|+2}(t', \underline{u}) dt'. \end{aligned} \quad (8.18)$$

Similarly to the contributions associated to K_1 , these contributions are bounded by (up to a constant)

$$\delta^{-1/2} \int_0^{\underline{u}} \underline{F}_{\leq |\alpha|+2}(t, \underline{u}') d\underline{u}' + \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \underline{E}_{\leq |\alpha|+2}(t', \underline{u}) dt' + \delta^{1/2} K_{\leq |\alpha|+2}(t, \underline{u}). \quad (8.19)$$

8.2. Top Order Optical Estimates. Now we estimate the contributions from the top order optical terms to the error spacetime integrals. In estimating the top order optical terms, we need to choose the power of $\mu_m(t)$ large enough. Therefore from this subsection on, we will use C to denote an absolute positive constant so that one can see the largeness of the power of $\mu_m(t)$ more clearly.

The top order optical terms come from the term in which all the commutators hit the deformation tensors in the expression of $^{(Z_1)}\sigma_1$, namely, the term:

$$(Z_{|\alpha|+1} + ^{(Z_{|\alpha|+1})}\delta) \dots (Z_2 + ^{(Z_2)}\delta) ^{(Z_1)}\sigma_1$$

more precisely, in:

$$Z_{|\alpha|+1} \dots Z_2 \left(-\frac{1}{2} L(^{(Z_1)}J_{1,\underline{L}}) - \frac{1}{2} \underline{L} (^{(Z_1)}J_{1,L}) + \text{div}(\mu^{(Z_1)}\sharp) \right)$$

when the operators $L, \underline{L}, \text{div}$ hit the deformation tensors in the expression of $^{(Z_1)}J$.

Now we consider the top order variations:

$$R_i^{\alpha+1}\psi, \quad R_i^{\alpha'} T^{l+1}\psi, \quad Q R_i^{\alpha'} T^l \psi$$

where $|\alpha| = N_{\text{top}} - 1$ and $|\alpha'| + l + 1 = |\alpha| + 1$. Then the corresponding principal optical terms are:

$$\begin{aligned} \tilde{\rho}_{|\alpha|+2}(R_i^{\alpha+1}\psi) &:= \frac{1}{c}(R_i^{\alpha+1}\text{tr}\underline{\chi}') \cdot T\psi, \quad \tilde{\rho}_{|\alpha|+2}(R_i^{\alpha'} T^{l+1}\psi) := \frac{1}{c}(R_i^{\alpha'} T^l \Delta\mu) \cdot T\psi \\ \tilde{\rho}_{|\alpha|+2}(Q R_i^{\alpha'} T^l \psi) &:= \frac{t\mu}{c}(\sharp R_i^{\alpha'} \text{tr}\underline{\chi}') \cdot \sharp\psi + \frac{t\mu}{c}(\sharp R_i^{\alpha'} \Delta\mu) \cdot \underline{L}\psi, \quad \text{if } l = 0 \\ \tilde{\rho}_{|\alpha|+2}(Q R_i^{\alpha'} T^l \psi) &:= \frac{t\mu}{c}(\sharp R_i^{\alpha'} T^{l-1} \Delta\mu) \cdot \sharp\psi + \frac{t\mu}{c}(\sharp R_i^{\alpha'} T^l \Delta\mu) \cdot \underline{L}\psi, \quad \text{if } l \geq 1 \end{aligned}$$

Here we used the structure equation (2.32)

$$\sharp_T \text{tr}\underline{\chi} = \Delta\mu + \mathcal{O}_{\geq 0,2}^{\leq 1}$$

Now we briefly investigate the behavior of the above terms with respect to δ and t . Note that $R_i^\alpha Q\psi$ has the same behavior as $R_i^{\alpha+1}\psi$ with respect to δ and t , while for their corresponding top order optical terms $\frac{t\mu}{c}(\sharp R_i^\alpha \text{tr}\underline{\chi}') \cdot \sharp\psi$ and $\frac{1}{c}(R_i^{\alpha+1}\text{tr}\underline{\chi}') \cdot T\psi$, the former behaves better than the latter with respect to δ and t :

$$|\frac{t\mu}{c}(\sharp R_i^\alpha \text{tr}\underline{\chi}') \cdot \sharp\psi| \sim |\mu(R_i^{\alpha+1}\text{tr}\underline{\chi}')| \delta^{1/2} (-t)^{-1}, \quad |\frac{1}{c}(R_i^{\alpha+1}\text{tr}\underline{\chi}') \cdot T\psi| \sim |(R_i^{\alpha+1}\text{tr}\underline{\chi}')| \delta^{-1/2}$$

We see that not only the former behaves better with respect to δ , but also has an extra μ , which makes the behavior even better when μ is small. This means that we only need to estimate the contribution of $\frac{1}{c}(R_i^{\alpha+1}\text{tr}\underline{\chi}') \cdot T\psi$. The same analysis applies to the terms involving $\underline{L}\psi$ as well as the comparison between $R_i^{\alpha'} T\psi$ and $Q R_i^{\alpha'} T\psi$, which correspond to $\frac{1}{c}(R_i^{\alpha'} \Delta\mu) \cdot T\psi$ and $\frac{t\mu}{c}(R_i^{\alpha'+1} \Delta\mu) \cdot \sharp\psi$. So in the following, we do not need to estimate the contributions corresponding to the variations containing a Q .

8.2.1. Contribution of K_0 . In this subsection we first estimate the spacetime integral:

$$\begin{aligned} \int_{W_{\underline{u}}^t} \frac{1}{c} |R_i^{\alpha+1}\text{tr}\underline{\chi}'| |T\psi| |L R_i^{\alpha+1}\psi| dt' d\underline{u}' d\mu_g &\lesssim \\ \int_{-r_0}^t \sup_{\Sigma_{t'}^{\underline{u}}} (\mu^{-1} |T\psi|) (-t') &\| \mu \sharp R_i^\alpha \text{tr}\underline{\chi}' \|_{L^2(\Sigma_{t'}^{\underline{u}})} \| L R_i^{\alpha+1}\psi \|_{L^2(\Sigma_{t'}^{\underline{u}})} dt'. \end{aligned} \tag{8.20}$$

Here we do not include the contribution from $\underline{L} R_i^{\alpha+1}\psi$ in the definition of K_0 , because compared to the spacetime error integral associated to K_1 , this contribution is lower order with respect to the behavior in t . We will see how this contribution is bounded when we estimate the spacetime integral for K_1 .

By (7.44), we have, in view of the monotonicity of $\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})$ in t :

$$\begin{aligned}
(-t) \|\mathcal{L} R_i^\alpha \text{tr} \chi\|_{L^2(\Sigma_t^{\underline{u}})} &\leq C \delta \int_{-r_0}^t (-t')^{-2} \mathcal{B}_{0, \leq|\alpha|+1}(-r_0) dt' \\
&\quad + C \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) (-t)^{-1} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\
&\quad + C \delta^{1/2} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t') dt' \\
&\quad + C \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\
&\quad + C \delta^{-1/2} \mu_m^{-b_{|\alpha|+2}}(t) (-t)^{-1} \left(\int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \right)^{1/2}.
\end{aligned} \tag{8.21}$$

Without loss of generality, here we assume that there is a $t_0 \in [-r_0, t^*)$ such that $\mu_m(t_0) = \frac{1}{10}$ and $\mu_m(t') \geq \frac{1}{10}$ for $t \leq t_0$. If there is no such t_0 in $[-r_0, t^*)$, then $\mu_m(t)$ has an absolute positive lower bound for all $[-r_0, t^*)$ and it is clear to see that the following argument simplifies and also works in this case. In view of part (1') and part (2) in Lemma 7.1 and the fact $\mu_m(t') \geq \frac{1}{10}$ for $t' \in [-r_0, t_0]$, we have

$$\begin{aligned}
\int_{-r_0}^{t_0} \mu_m^{-b_{|\alpha|+2}-1/2}(t') dt' &\lesssim \mu_m^{-b_{|\alpha|+2}+1/2}(t_0) \leq \mu_m^{-b_{|\alpha|+2}+1/2}(t), \\
\int_{t_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') dt' &\lesssim \frac{1}{(b_{|\alpha|+2} - 1/2)} \mu_m^{-b_{|\alpha|+2}+1/2}(t).
\end{aligned}$$

Therefore the third term in (8.21) are bounded by:

$$\delta^{1/2} \mu_m^{-b_{|\alpha|+2}+1/2}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})}.$$

Substituting this in (8.20), and using the fact that $|T\psi| \lesssim \delta^{-1/2}(-t)^{-1}$, we see that the spacetime integral (8.20) is bounded by (up to a constant):

$$\begin{aligned}
&\int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}-1}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \|LR_i^{\alpha+1}\psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \\
&+ \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \|LR_i^{\alpha+1}\psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \\
&+ \int_{-r_0}^t \delta^{-1} (-t')^{-2} \mu_m^{-b_{|\alpha|+2}-1}(t') \sqrt{\int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}') d\underline{u}'} \|LR_i^{\alpha+1}\psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \\
&+ \int_{-r_0}^t (-t')^{-1} \mu_m^{-b_{|\alpha|+2}-1}(t') \left(\int_{-r_0}^{t'} (-t'')^{-2} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t'', \underline{u})} dt'' \right) \|LR_i^{\alpha+1}\psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt'.
\end{aligned} \tag{8.22}$$

For the factor $\|LR_i^{\alpha+1}\psi\|_{L^2(\Sigma_t^{\underline{u}})}$, we bound it as:

$$\|LR_i^{\alpha+1}\psi\|_{L^2(\Sigma_t^{\underline{u}})} \leq \sqrt{E_{|\alpha|+2}(t, \underline{u})} \leq \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})},$$

To estimate (8.22) we split the integral as $\int_{-r_0}^{t_0} + \int_{t_0}^t$, which we call the “non-shock” and “shock” parts respectively. In view of the second part of Lemma 7.1, the “non-shock” part of (8.22) is bounded by (up to a constant)

$$\begin{aligned}
& \mu_m^{-2b_{|\alpha|+2}+1/2}(t) \int_{-r_0}^{t_0} (-t')^{-2} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' \\
& + \delta^{-2} \mu_m^{-2b_{|\alpha|+2}}(t) \int_0^{\underline{u}} \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \\
& + \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^{t_0} (-t')^{-2} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt'
\end{aligned} \tag{8.23}$$

Following the proof of Lemma 7.1, we have, for any $a > 0$

$$\int_{t_0}^t \mu_m^{-a-1}(t') (-t')^{-2} dt' \leq \frac{C}{a} \mu_m^{-a}(t). \tag{8.24}$$

Therefore the “shock part” of (8.22) is bounded by (up to a constant)

$$\boxed{\frac{\mu_m^{-2b_{|\alpha|+2}}(t)}{2b_{|\alpha|+2}} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} + \frac{\mu_m^{-2b_{|\alpha|+2}+1/2}(t)}{2b_{|\alpha|+2} - 1/2} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \frac{\delta^{-2} \mu_m^{-2b_{|\alpha|+2}}(t)}{2b_{|\alpha|+2}} \int_0^{\underline{u}} \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}'. \tag{8.25}$$

Remark 8.2. The boxed term is from the estimates for the top order term F_α . In view of (7.3), the number of top order terms contributed by the variations is independent of δ and $|\alpha|$, so is the constant C in the boxed term. Later on in the top order energy estimates we will choose $b_{|\alpha|+2}$ in such a way that $\frac{C}{b_{top}} \leq \frac{1}{10}$. (The purpose of doing this is to make sure that this term can be absorbed by the left hand side the energy inequality.) Therefore we can choose $b_{top} = \{10, \frac{C}{20}\}$. In particular b_{top} is independent of δ .

Next we consider the spacetime integral:

$$\begin{aligned}
& \delta^{2l+2} \int_{W_{\underline{u}}^t} \frac{1}{c} |R_i^{\alpha'} T^l \Delta \mu| |T\psi| |LR^{\alpha'} T^{l+1} \psi| dt' d\underline{u}' d\mu_g \lesssim \\
& \delta^{2l+2} \int_{-r_0}^t \sup_{\Sigma_{t'}^{\underline{u}}} (\mu^{-1} |T\psi|) \|\mu R_i^{\alpha'} T^l \Delta \mu\|_{L^2(\Sigma_{t'}^{\underline{u}})} \|LR^{\alpha'} T^{l+1} \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt'.
\end{aligned} \tag{8.26}$$

In view of (7.64) and Lemma 7.1 we have

$$\begin{aligned}
& \delta^{l+1} \|\mu R^{\alpha'} T^l \Delta \mu\|_{L^2(\Sigma_t^{\underline{u}})} \lesssim \delta^{l+1} r_0 (-t)^{-1} \|F'_{\alpha, l}(-r_0)\|_{L^2(\Sigma_{-r_0}^{\underline{u}})} \\
& + \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) (-t)^{-1} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\
& + \delta^{3/2} (-t)^{-2} \mu_m^{-b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \\
& + \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) \int_{-r_0}^t (-t')^{-2} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\
& + \delta^{1/2} \mu_m^{-b_{|\alpha|+2}+1/2}(t) \int_{-r_0}^t (-t')^{-2} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\
& + \delta^{1/2} \mu_m^{-b_{|\alpha|+2}}(t) (-t)^{-1} \left(\int_0^{\underline{u}} \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \right)^{1/2}.
\end{aligned} \tag{8.27}$$

Substituting this to (8.26) and using the fact that $|T\psi| \lesssim \delta^{-1/2} (-t)^{-1}$, we see that (8.26) is bounded by (up to a constant)

$$\begin{aligned}
& \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}-1}(t')(-t')^{-2} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' \\
& + \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}-1}(t')(-t')^{-3} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\
& + \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}-1}(t')(-t')^{-1} \int_{-r_0}^{t'} (-t'')^{-2} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt'' \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\
& + \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}-1}(t')(-t')^{-2} \left(\int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}') d\underline{u}' \right)^{1/2} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt'.
\end{aligned} \tag{8.28}$$

As before we split the spacetime integral (8.28) as the “non-shock” and “shock” parts. In view of Lemma 7.1, the “non-shock” part is bounded by (up to a constant)

$$\begin{aligned}
& \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^{t_0} (-t')^{-2} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' \\
& + \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^{t_0} (-t')^{-3} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' \\
& + \mu_m^{-2b_{|\alpha|+2}}(t) \int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}') d\underline{u}'.
\end{aligned} \tag{8.29}$$

The “shock” part is bounded by (up to a constant)

$$\frac{\mu_m^{-2b_{|\alpha|+2}}(t)}{2b_{|\alpha|+2}} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \frac{\mu_m^{-2b_{|\alpha|+2}}(t)}{2b_{|\alpha|+2}} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \frac{\mu_m^{-2b_{|\alpha|+2}}(t)}{2b_{|\alpha|+2}} \int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}'. \tag{8.30}$$

8.2.2. *Contribution of K_1 .* We now turn to estimate the contributions of top order optical terms associated to K_1 . Let us start with the following absolute value of a spacetime integral:

$$\left| \int_{W_{\underline{u}}^t} \frac{1}{c} \frac{2(-t)}{\widetilde{\text{tr}} \underline{\chi}} (R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot (T\psi) \cdot (\underline{L} R_i^{\alpha+1} \psi + \frac{1}{2} \widetilde{\text{tr}} \underline{\chi} R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\underline{g}} \right|. \tag{8.31}$$

In view of the relation between \underline{g} and \tilde{g} , the above integral can be written as

$$\int_{W_{\underline{u}}^t} \frac{2(-t)}{\widetilde{\text{tr}} \underline{\chi}} (R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot (T\psi) \cdot (\underline{L} R_i^{\alpha+1} \psi + \frac{1}{2} \widetilde{\text{tr}} \underline{\chi} R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}}. \tag{8.32}$$

For a smooth function f we have

$$\frac{\partial}{\partial t} \left(\int_{S_{t, \underline{u}}} f d\mu_{\tilde{g}} \right) = \int_{S_{t, \underline{u}}} (\underline{L} f + \widetilde{\text{tr}} \underline{\chi} f) d\mu_{\tilde{g}},$$

which implies

$$\int_{W_{\underline{u}}^t} (\underline{L} f + \widetilde{\text{tr}} \underline{\chi} f) d\mu_{\tilde{g}} d\underline{u}' dt' = \int_{\Sigma_t^{\underline{u}}} f d\mu_{\tilde{g}} d\underline{u}' - \int_{\Sigma_{-r_0}^{\underline{u}}} f d\mu_{\tilde{g}} d\underline{u}'.$$

This inspires us to write the spacetime integral (8.32) as

$$\begin{aligned}
& \int_{W_{\underline{u}}^t} (R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot \left(\frac{2(-t')}{\widetilde{\text{tr} \underline{\chi}}} T\psi \right) \cdot (\underline{L} R_i^{\alpha+1} \psi + \widetilde{\text{tr} \underline{\chi}} R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
& - \frac{1}{2} \int_{W_{\underline{u}}^t} \widetilde{\text{tr} \underline{\chi}} (R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot \left(\frac{2(-t')}{\widetilde{\text{tr} \underline{\chi}}} T\psi \right) \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
& = \int_{W_{\underline{u}}^t} (\underline{L} + \widetilde{\text{tr} \underline{\chi}}) \left[(R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot \left(\frac{2(-t')}{\widetilde{\text{tr} \underline{\chi}}} T\psi \right) \cdot (R_i^{\alpha+1} \psi) \right] dt' d\underline{u}' d\mu_{\tilde{g}} \\
& - \int_{W_{\underline{u}}^t} (\underline{L} + \frac{1}{2} \widetilde{\text{tr} \underline{\chi}}) \left[(R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot \left(\frac{2(-t')}{\widetilde{\text{tr} \underline{\chi}}} T\psi \right) \right] \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}}.
\end{aligned}$$

We conclude that (8.32) equals:

$$\begin{aligned}
& \int_{\Sigma_{\underline{u}}^t} (R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot \left(\frac{2(-t)}{\widetilde{\text{tr} \underline{\chi}}} T\psi \right) \cdot (R_i^{\alpha+1} \psi) d\underline{u}' d\mu_{\tilde{g}} \\
& - \int_{\Sigma_{\underline{u}}^t} (R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot \left(\frac{2r_0}{\widetilde{\text{tr} \underline{\chi}}} T\psi \right) \cdot (R_i^{\alpha+1} \psi) d\underline{u}' d\mu_{\tilde{g}} \\
& - \int_{W_{\underline{u}}^t} (\underline{L} + \frac{1}{2} \widetilde{\text{tr} \underline{\chi}}) \left[(R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot \left(\frac{2(-t')}{\widetilde{\text{tr} \underline{\chi}}} T\psi \right) \right] \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}}.
\end{aligned}$$

We will be using the following lemma for integration by parts:

Lemma 8.3. *Let f, g be arbitrary smooth functions defined on $S_{t, \underline{u}}$ and X be a vectorfield tangent to $S_{t, \underline{u}}$. We have:*

$$\int_{S_{t, \underline{u}}} f(Xg) d\mu_{\tilde{g}} = - \int_{S_{t, \underline{u}}} g(Xf) d\mu_{\tilde{g}} - \frac{1}{2} \int_{S_{t, \underline{u}}} \text{tr}^{(X)} \tilde{\not{r}} d\mu_{\tilde{g}}.$$

We first consider the hypersurface integral which, integrating by parts using the above lemma, equals:

$$-H_0 - H_1 - H_2$$

where:

$$\begin{aligned}
H_0 &= \int_{\Sigma_{\underline{u}}^t} (R_i^{\alpha} \text{tr} \underline{\chi}) \cdot \left(\frac{2(-t')}{\widetilde{\text{tr} \underline{\chi}}} T\psi \right) \cdot (R_i^{\alpha+2} \psi) d\underline{u}' d\mu_{\tilde{g}} \\
H_1 &= \int_{\Sigma_{\underline{u}}^t} \frac{2(-t')}{\widetilde{\text{tr} \underline{\chi}}} (R_i^{\alpha} \text{tr} \underline{\chi}') \cdot (R_i T\psi) \cdot (R_i^{\alpha+1} \psi) d\underline{u}' d\mu_{\tilde{g}} \\
H_2 &= \int_{\Sigma_{\underline{u}}^t} (T\psi) \cdot (R_i^{\alpha+1} \psi) \cdot (R^{\alpha} \text{tr} \underline{\chi}') \left(2R_i \left(\frac{(-t')}{\widetilde{\text{tr} \underline{\chi}}} \right) + \frac{1}{2} \text{tr}^{(R_i)} \tilde{\not{r}} \right) d\underline{u}' d\mu_{\tilde{g}}.
\end{aligned}$$

Since we bound both $T\psi$ and $R_i T\psi$ in L^∞ norm, compared to H_0 , H_1 is a lower order term with respect to the order of derivatives. While for H_2 , we use the estimate:

$$2 \left| R_i \left(\frac{(-t')}{\widetilde{\text{tr} \underline{\chi}}} \right) \right| + |\text{tr}^{(R_i)} \tilde{\not{r}}| \lesssim \delta$$

to see that it is a lower order term with respect to both the behavior of $\delta, (-t')$ and the order of derivatives compared to H_0 . This analysis tells us that we only need to estimate H_0 .

$$\begin{aligned}
|H_0| &\leq \int_{\Sigma_t^u} (-t)^3 |T\psi| \|R_i^\alpha \text{tr}\underline{\chi}'\| \|\not{d}R_i^{\alpha+1}\psi\| d\underline{u}' d\mu_{\not{g}} \leq \delta^{-1/2} (-t)^2 \|R_i^\alpha \text{tr}\underline{\chi}'\|_{L^2(\Sigma_t^u)} \|\not{d}R_i^{\alpha+1}\psi\|_{L^2(\Sigma_t^u)} \\
&\lesssim \delta^{-1/2} (-t) \|R_i^\alpha \text{tr}\underline{\chi}'\|_{L^2(\Sigma_t^u)} \mu_m^{-1/2}(t) \sqrt{E_{\leq |\alpha|+2}(t, \underline{u})} \\
&\lesssim \delta^{-1/2} (-t) \|R_i^\alpha \text{tr}\underline{\chi}'\|_{L^2(\Sigma_t^u)} \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})}.
\end{aligned} \tag{8.33}$$

Even though Proposition 6.3 gives an L^2 -estimate for $R_i^\alpha \text{tr}\underline{\chi}'$, here we give an alternative proof which will be used later. In view of (2.33) and the relation (3.18) we have

$$\underline{L} \text{tr}\underline{\chi}' - \frac{2}{\underline{u} - t} \text{tr}\underline{\chi}' = e \text{tr}\underline{\chi} - |\underline{\chi}'|_{\not{g}}^2 - \text{tr}\alpha' := \rho_0. \tag{8.34}$$

Applying R_i^α to this equation we have

$$\underline{L} R_i^\alpha \text{tr}\underline{\chi}' - \frac{2}{\underline{u} - t} R_i^\alpha \text{tr}\underline{\chi}' = R_i^\alpha \rho_0 + \sum_{|\beta| \leq |\alpha|} R_i^{\alpha-\beta(R_i)} \underline{Z} R_i^{\beta-1} \text{tr}\underline{\chi}', \tag{8.35}$$

which can be rewritten as

$$\underline{L} ((t - \underline{u})^2 R_i^\alpha \text{tr}\underline{\chi}') = (t - \underline{u})^2 \left(R_i^\alpha \rho_0 + \sum_{|\beta| \leq |\alpha|} R_i^{\alpha-\beta(R_i)} \underline{Z} R_i^{\beta-1} \text{tr}\underline{\chi}' \right) := \rho_\alpha. \tag{8.36}$$

In view of

$$\|^{(R_i)} \underline{Z}\|_{L^\infty(\Sigma_t^u)} \lesssim \delta(-t)^{-2}$$

and Proposition 6.3 we have

$$\begin{aligned}
\|(t - \underline{u})^2 R_i^{\alpha-\beta(R_i)} \underline{Z} R_i^{\beta-1} \text{tr}\underline{\chi}'\|_{L^2(\Sigma_t^u)} &\lesssim \delta^{3/2} \int_{-r_0}^t (-t')^{-3} \mu_m^{-1/2}(t') \sqrt{E_{\leq |\alpha|+2}(t', \underline{u})} dt' \\
&\lesssim \delta^{3/2} \int_{-r_0}^t (-t')^{-3} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt' \\
&\lesssim \delta^{3/2} (-t)^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})}.
\end{aligned} \tag{8.37}$$

Also by Proposition 6.1, 6.3, the contribution of the second term in ρ_0 to the $L^2(\Sigma_t^u)$ -norm of ρ_α is also bounded by (8.37). In view of the definition of e the contribution of the first term in ρ_0 to the $L^2(\Sigma_t^u)$ -norm of ρ_α is bounded by

$$\delta^{1/2} (-t)^{-2} \mu_m^{-1/2}(t) \sqrt{E_{\leq |\alpha|+2}(t, \underline{u})} \lesssim \delta^{1/2} (-t)^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})}. \tag{8.38}$$

In view of the definition of α' , the $L^2(\Sigma_t^u)$ -norm of other contributions of ρ_0 to ρ_α is bounded by

$$\delta^{1/2} (-t)^{-1} \mu_m^{-1/2}(t) \sqrt{E_{\leq |\alpha|+2}(t, \underline{u})} \lesssim \delta^{1/2} (-t)^{-1} \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})}. \tag{8.39}$$

Integrating the propagation equation (8.36) we have

$$\begin{aligned}
& (-t) \|R_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_t^\underline{u})} \lesssim (-t)^2 \|R_i^\alpha \text{tr} \underline{\chi}'(t)\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} \\
& \lesssim \int_{-r_0}^t \|\rho_\alpha(t')\|_{L^2([0, \underline{u}] \times \mathbb{S}^2)} dt' \lesssim \int_{-r_0}^t (-t')^{-1} \|\rho_\alpha\|_{L^2(\Sigma_{t'}^\underline{u})} dt' \\
& \lesssim \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt'.
\end{aligned} \tag{8.40}$$

Substituting this in (8.33) $|H_0|$ is bounded by

$$\mu_m^{-2b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt'. \tag{8.41}$$

As before, we consider the “shock part $\int_{-r_0}^{t_0}$ ” and “non-shock part $\int_{t_0}^t$ ”, which are denoted by H_0^S and H_0^N , separately.

For $t' \in [-r_0, t_0]$ we have $\mu_m^{-1}(t') \leq 10$. Therefore the time integral in the “non-shock” part is bounded by

$$\begin{aligned}
& \int_{-r_0}^{t_0} \mu_m^{-b_{|\alpha|+2}-1/2}(t') (-t')^{-2} \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt' \\
& = \int_{-r_0}^{t_0} \mu_m^{-1}(t') \mu_m^{-b_{|\alpha|+2}+1/2}(t') (-t')^{-2} \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt' \\
& \lesssim \int_{-r_0}^{t_0} \mu_m^{-b_{|\alpha|+2}+1/2}(t') (-t')^{-2} \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt' \\
& \lesssim \int_{-r_0}^{t_0} (-t')^{-2} \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt' \cdot \mu_m^{-b_{|\alpha|+2}+1/2}(t).
\end{aligned}$$

Here in the last step we used Lemma 7.1. Therefore, using Holder’s inequality we have

$$\begin{aligned}
|H_0^N| & \leq C \int_{-r_0}^{t_0} (-t')^{-2} \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt' \cdot \mu_m^{-2b_{|\alpha|+2}}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \\
& \leq \epsilon \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + C_\epsilon \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^{t_0} (-t')^{-2} \tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) dt'
\end{aligned}$$

For the “shock part”, by the monotonicity of $\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})$ we have

$$\begin{aligned}
|H_0^S| & \lesssim \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \int_{t_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') (-t')^{-2} dt' \cdot \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \\
& \lesssim \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) \mu_m^{-b_{|\alpha|+2}-1/2}(t) \int_{t_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') (-t')^{-2} dt' \\
& \lesssim \frac{1}{(b_{|\alpha|+2} - 1/2)} \mu_m^{-b_{|\alpha|+2}-1/2}(t) \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) \cdot \mu_m^{-b_{|\alpha|+2}+1/2}(t) \\
& = \frac{1}{(b_{|\alpha|+2} - 1/2)} \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}).
\end{aligned}$$

We obtain the following estimate for $|H_0|$:

$$\begin{aligned}
|H_0| &\leq \frac{C}{(b_{|\alpha|+2} - 1/2)} \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) \\
&\quad + C_\epsilon \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^{t_0} (-t')^{-2} \tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) dt' + \epsilon \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}).
\end{aligned} \tag{8.42}$$

Next we consider the spacetime integral:

$$\begin{aligned}
&\int_{W_{\underline{u}}^t} (\underline{L} + \frac{1}{2} \tilde{\text{tr}} \underline{\chi}) \left[(R_i^{\alpha+1} \text{tr} \underline{\chi}) \cdot \left(\frac{2(-t')}{\tilde{\text{tr}} \underline{\chi}} T\psi \right) \right] \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
&= \int_{W_{\underline{u}}^t} ((\underline{L} + \tilde{\text{tr}} \underline{\chi})(R_i^{\alpha+1} \text{tr} \underline{\chi}')) \cdot \left(\frac{2(-t')}{\tilde{\text{tr}} \underline{\chi}} T\psi \right) \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
&\quad + \int_{W_{\underline{u}}^t} (R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot (\underline{L} + \frac{1}{2} \tilde{\text{tr}} \underline{\chi}) \left(\frac{2(-t')}{\tilde{\text{tr}} \underline{\chi}} T\psi \right) \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
&\quad - \int_{W_{\underline{u}}^t} (\tilde{\text{tr}} \underline{\chi})(R_i^{\alpha+1} \text{tr} \underline{\chi}') \cdot \left(\frac{2(-t')}{\tilde{\text{tr}} \underline{\chi}} T\psi \right) \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} := I + II + III.
\end{aligned}$$

We start with III . Using Lemma 8.3 we rewrite III as

$$\begin{aligned}
&\int_{W_{\underline{u}}^t} (\tilde{\text{tr}} \underline{\chi})(R_i^{\alpha} \text{tr} \underline{\chi}') \cdot \left(\frac{2(-t')}{\tilde{\text{tr}} \underline{\chi}} T\psi \right) \cdot (R_i^{\alpha+2} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
&+ \int_{W_{\underline{u}}^t} (R_i \tilde{\text{tr}} \underline{\chi})(R_i^{\alpha} \text{tr} \underline{\chi}') \cdot \left(\frac{2(-t')}{\tilde{\text{tr}} \underline{\chi}} T\psi \right) \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
&+ \int_{W_{\underline{u}}^t} (\tilde{\text{tr}} \underline{\chi})(R_i^{\alpha} \text{tr} \underline{\chi}') \cdot R_i \left(\frac{2(-t')}{\tilde{\text{tr}} \underline{\chi}} T\psi \right) \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
&+ \frac{1}{2} \int_{W_{\underline{u}}^t} \text{tr}^{(R_i)} \tilde{\not{\chi}} (\tilde{\text{tr}} \underline{\chi})(R_i^{\alpha} \text{tr} \underline{\chi}') \cdot \left(\frac{2(-t')}{\tilde{\text{tr}} \underline{\chi}} T\psi \right) \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
&=: III_1 + III_2 + III_3 + III_4.
\end{aligned}$$

Compared to III_3 , III_2 and III_4 are lower order due to the factor $R_i \tilde{\text{tr}} \underline{\chi} + \frac{1}{2} \text{tr}^{(R_i)} \tilde{\not{\chi}}$. While III_3 is lower order compared to III_1 since ψ receives less derivatives. So we only need to estimate III_1 . In view of 8.40 III_1 can be bounded as

$$\begin{aligned}
& \delta^{-1/2} \int_{-r_0}^t (-t') \|R_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_{t'}^u)} \|\not{d} R_i^{\alpha+1} \psi\|_{L^2(\Sigma_{t'}^u)} dt' \\
& \lesssim \int_{-r_0}^t \left(\int_{-r_0}^{t'} (-t'')^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t'') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t'', \underline{u})} dt'' \right) \\
& \quad \mu_m^{-b_{|\alpha|+2}-1/2}(t') (-t')^{-1} \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} dt' \\
& \lesssim \int_{-r_0}^t (-t')^{-2} \mu_m^{-2b_{|\alpha|+2}-1}(t') \tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) dt' \\
& \lesssim \frac{1}{2b_{|\alpha|+2}} \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^{t_0} (-t')^{-2} \tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) dt'.
\end{aligned} \tag{8.43}$$

As before, here we split the time integral into the “shock” and “non-shock” parts and use Lemma 7.1.

Now let us move to *II*. Note that the factor involving $T\psi$ can be rewritten as

$$\frac{2(-t')}{\widetilde{\text{tr} \underline{\chi}}} (\underline{L} + \frac{1}{2} \widetilde{\text{tr} \underline{\chi}})(T\psi) - \frac{2(-t')}{(\widetilde{\text{tr} \underline{\chi}})^2} \underline{L}(\widetilde{\text{tr} \underline{\chi}}) T\psi := T_1 + T_2.$$

By the equation (2.35), the contribution of T_2 to *II* is lower order with respect to δ compared to *III*. On the other hand, the equation (4.1) implies the pointwise estimate

$$\|(\underline{L} + \frac{1}{2} \widetilde{\text{tr} \underline{\chi}})(T\psi)\|_{L^\infty(\Sigma_t^u)} \lesssim \delta^{1/2} (-t)^{-3}, \tag{8.44}$$

which shows that the contribution from T_1 to *II* is also lower order with respect to δ and t compared to *III*.

For *I*, we first note that

$$(\underline{L} + \widetilde{\text{tr} \underline{\chi}})(R_i^{\alpha+1} \text{tr} \underline{\chi}') = R_i(\underline{L} + \widetilde{\text{tr} \underline{\chi}})(R_i^\alpha \text{tr} \underline{\chi}') + {}^{(R_i)}\underline{Z} R_i^\alpha \text{tr} \underline{\chi}' - R_i(\text{tr} \underline{\chi}') R_i^\alpha \text{tr} \underline{\chi}' + \text{l.o.t.} \tag{8.45}$$

The lower order term above is lower order compared to the second term above. By the pointwise estimate for ${}^{(R_i)}\underline{Z}$ and Proposition 6.1, the contributions of the second and the third term are lower order with respect to δ and t compared to *III*.

For the contribution of the first term in (8.45), we use Lemma 8.3 to write it as

$$\begin{aligned}
& - \int_{W_{\underline{u}}^t} ((\underline{L} + \widetilde{\text{tr} \underline{\chi}})(R_i^\alpha \text{tr} \underline{\chi}')) \cdot R_i \left(\frac{2(-t')}{\widetilde{\text{tr} \underline{\chi}}} T\psi \right) \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
& - \int_{W_{\underline{u}}^t} ((\underline{L} + \widetilde{\text{tr} \underline{\chi}})(R_i^\alpha \text{tr} \underline{\chi}')) \cdot \left(\frac{2(-t')}{\widetilde{\text{tr} \underline{\chi}}} T\psi \right) \cdot (R_i^{\alpha+2} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
& - \int_{W_{\underline{u}}^t} \text{tr} {}^{(R_i)}\widetilde{\not{d}}((\underline{L} + \widetilde{\text{tr} \underline{\chi}})(R_i^\alpha \text{tr} \underline{\chi}')) \cdot \left(\frac{2(-t')}{\widetilde{\text{tr} \underline{\chi}}} T\psi \right) \cdot (R_i^{\alpha+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
& = -I_{11} - I_{12} - I_{13}.
\end{aligned} \tag{8.46}$$

Again, due to the pointwise estimates for $R_i \text{tr} \underline{\chi}$ and $\text{tr} {}^{(R_i)}\widetilde{\not{d}}$, I_{11} and I_{13} are lower order with respect to δ and t compared to I_{12} . By the equation (8.35),

$$(\underline{L} + \widetilde{\text{tr}}\underline{\chi})(R_i^{\alpha'} \text{tr}\underline{\chi}') = \left(\underline{L} + \frac{2}{t - \underline{u}} \right) (R_i^{\alpha'} \text{tr}\underline{\chi}') + \text{l.o.t.} = \frac{\rho_{\alpha}}{(t - \underline{u})^2} + \text{l.o.t.}.$$

Here l.o.t. is bounded as

$$\|\text{l.o.t.}\|_{L^2(\Sigma_{t'}^{\underline{u}})} \lesssim \delta(-t')^{-3} \|R_i^{\alpha'} \text{tr}\underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})}, \quad (8.47)$$

whose contribution to I_{12} is lower order compared to III_1 . In view of (8.37)-(8.39),

$$\left\| \frac{\rho_{\alpha}}{(t - \underline{u})^2} \right\|_{L^2(\Sigma_{t'}^{\underline{u}})} \lesssim \delta^{1/2}(-t')^{-3} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})}. \quad (8.48)$$

Thus the principal contribution in I_{12} is bounded by

$$\begin{aligned} & \int_{-r_0}^t (-t')^{-2} \mu_m^{-2b_{|\alpha|+2}-1}(t') \widetilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' \\ & \lesssim \frac{1}{2b_{|\alpha|+2}} \mu_m^{-2b_{|\alpha|+2}}(t) \widetilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^{t_0} (-t')^{-2} \widetilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt'. \end{aligned} \quad (8.49)$$

This completes the estimate for the spacetime integral (8.32).

Next we consider the top order optical contribution of the variation $R^{\alpha'} T^{l+1} \psi$, where $|\alpha'| + l + 1 = |\alpha| + 1$, which is the following spacetime integral:

$$\delta^{2l+2} \left| \int_{W_{\underline{u}}^t} \frac{2(-t')}{\widetilde{\text{tr}}\underline{\chi}} (T\psi) \cdot (R_i^{\alpha'} T^l \underline{\Delta}\mu) \cdot \left((\underline{L} + \frac{1}{2} \widetilde{\text{tr}}\underline{\chi}) R_i^{\alpha'} T^{l+1} \psi \right) dt' d\underline{u}' d\mu_{\tilde{g}} \right| \quad (8.50)$$

Again, we rewrite the above spacetime integral as:

$$\begin{aligned} & \delta^{2l+2} \int_{W_{\underline{u}}^t} \frac{2(-t')}{\widetilde{\text{tr}}\underline{\chi}} (T\psi) \cdot (R_i^{\alpha'} T^l \underline{\Delta}\mu) \cdot \left((\underline{L} + \widetilde{\text{tr}}\underline{\chi}) (R_i^{\alpha'} T^{l+1} \psi) \right) dt' d\underline{u}' d\mu_{\tilde{g}} \\ & - \delta^{2l+2} \int_{W_{\underline{u}}^t} \frac{(-t')}{\widetilde{\text{tr}}\underline{\chi}} (T\psi) \cdot (R_i^{\alpha'} T^l \underline{\Delta}\mu) \cdot (\widetilde{\text{tr}}\underline{\chi} (R_i^{\alpha'} T^{l+1} \psi)) dt' d\underline{u}' d\mu_{\tilde{g}} \end{aligned}$$

which is:

$$\begin{aligned} & \delta^{2l+2} \int_{W_{\underline{u}}^t} (\underline{L} + \widetilde{\text{tr}}\underline{\chi}) \left(\frac{2(-t')}{\widetilde{\text{tr}}\underline{\chi}} (T\psi) (R_i^{\alpha'} T^l \underline{\Delta}\mu) (R_i^{\alpha'} T^{l+1} \psi) \right) dt' d\underline{u}' d\mu_{\tilde{g}} \\ & - \delta^{2l+2} \int_{W_{\underline{u}}^t} \left((\underline{L} + \frac{1}{2} \widetilde{\text{tr}}\underline{\chi}) \left(\frac{2(-t')}{\widetilde{\text{tr}}\underline{\chi}} T\psi \right) \right) (R_i^{\alpha'} T^l \underline{\Delta}\mu) (R_i^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\ & - \delta^{2l+2} \int_{W_{\underline{u}}^t} \frac{2(-t')}{\widetilde{\text{tr}}\underline{\chi}} (T\psi) ((\underline{L} + \widetilde{\text{tr}}\underline{\chi}) (R_i^{\alpha'} T^l \underline{\Delta}\mu)) (R_i^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\ & + \delta^{2l+2} \int_{W_{\underline{u}}^t} \frac{2(-t')}{\widetilde{\text{tr}}\underline{\chi}} (T\psi) (\widetilde{\text{tr}}\underline{\chi} (R_i^{\alpha'} T^l \underline{\Delta}\mu)) (R_i^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\ & =: H' + I' + II' + III'. \end{aligned}$$

As before, the spacetime integral in the first line above can be written as:

$$\delta^{2l+2} \int_{\Sigma_t^u} \frac{2(-t)}{\widetilde{\text{tr}} \underline{\chi}} (T\psi)(R_i^{\alpha'} T^l \underline{\Delta} \mu)(R_i^{\alpha'} T^{l+1} \psi) d\underline{u}' d\mu_{\tilde{g}} - \delta^{2l+2} \int_{\Sigma_{-r_0}^u} \frac{2r_0}{\widetilde{\text{tr}} \underline{\chi}} (T\psi)(R_i^{\alpha'} T^l \underline{\Delta} \mu)(R_i^{\alpha'} T^{l+1} \psi) d\underline{u}' d\mu_{\tilde{g}}$$

The integral on Σ_t^u can be written as

$$\begin{aligned} & - \delta^{2l+2} \int_{\Sigma_t^u} \frac{2(-t)}{\widetilde{\text{tr}} \underline{\chi}} (T\psi)(R_i^{\alpha'-1} T^l \underline{\Delta} \mu)(R_i^{\alpha'+1} T^{l+1} \psi) d\underline{u}' d\mu_{\tilde{g}} \\ & - \delta^{2l+2} \int_{\Sigma_t^u} \left(R_i \left(\frac{2(-t)}{\widetilde{\text{tr}} \underline{\chi}} T\psi \right) + \frac{1}{2} \text{tr}^{(R_i)} \tilde{\not{r}} \right) (R_i^{\alpha'-1} T^l \underline{\Delta} \mu)(R_i^{\alpha'} T^{l+1} \psi) d\underline{u}' d\mu_{\tilde{g}} \\ & := -H'_0 - H'_1 \end{aligned}$$

In view of the estimates

$$\|\text{tr}^{(R_i)} \tilde{\not{r}}\|_{L^\infty(\Sigma_t^u)} \lesssim \delta(-t)^{-2}, \quad \|R_i \text{tr} \underline{\chi}\|_{L^\infty(\Sigma_t^u)} \lesssim \delta(-t)^{-3},$$

H'_1 is lower order with respect to δ and the order of derivatives compared to H'_0 . So we only estimate H'_0 . In view of Proposition 6.4, a preliminary estimate for H'_0 is given by

$$\begin{aligned} |H'_0| & \lesssim \delta^{-1/2+l+1}(-t) \|R_i^{\alpha'-1} T^l \underline{\Delta} \mu\|_{L^2(\Sigma_t^u)} \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \\ & \lesssim \delta \int_{-r_0}^t (-t')^{-2} \left(\mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} + (-t')^{-1} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} \right) dt' \\ & \quad \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})}. \end{aligned} \quad (8.51)$$

As before we split the time integral as “shock” and “non-shock” parts. The contribution from the “non-shock” part is bounded by

$$\begin{aligned} & \delta \mu_m^{-b_{|\alpha|+2}+1/2}(t) \int_{-r_0}^{t_0} (-t')^{-2} \left(\sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} + \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} \right) dt' \cdot \mu_m^{-b_{|\alpha|+2}-1/2}(t) \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \\ & \lesssim \delta \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}) + \delta \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^{t_0} (-t')^{-2} \left(\tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) + \tilde{E}_{\leq |\alpha|+2}(t', \underline{u}) \right) dt'. \end{aligned} \quad (8.52)$$

The contribution from the “shock” part is bounded by

$$\frac{\delta \mu_m^{-2b_{|\alpha|+2}+1/2}(t)}{b_{|\alpha|+2}-1} \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} \sqrt{\tilde{E}_{\leq |\alpha|+2}(t, \underline{u})} + \frac{\delta \mu_m^{-2b_{|\alpha|+2}}(t)}{b_{|\alpha|+2}-1/2} \tilde{E}_{\leq |\alpha|+2}(t, \underline{u}). \quad (8.53)$$

Next let us turn to the spacetime integrals I', II', III' . We start with III' . Using Lemma 8.3 III' can be written as

$$\begin{aligned}
& -\delta^{2l+2} \int_{W_{\underline{u}}^t} \frac{2(-t')}{\widetilde{\text{tr}}\underline{\chi}} (T\psi) (\widetilde{\text{tr}}\underline{\chi} (R_i^{\alpha'-1} T^l \underline{\Delta}\mu)) (R_i^{\alpha'+1} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
& -\delta^{2l+2} \int_{W_{\underline{u}}^t} \frac{2(-t')}{\widetilde{\text{tr}}\underline{\chi}} (T\psi) ((R_i \widetilde{\text{tr}}\underline{\chi}) (R_i^{\alpha'-1} T^l \underline{\Delta}\mu)) (R_i^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
& -\delta^{2l+2} \int_{W_{\underline{u}}^t} R_i \left(\frac{2(-t')}{\widetilde{\text{tr}}\underline{\chi}} (T\psi) \right) (\widetilde{\text{tr}}\underline{\chi} (R_i^{\alpha'-1} T^l \underline{\Delta}\mu)) (R_i^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
& -\delta^{2l+2} \int_{W_{\underline{u}}^t} \left(\text{tr}^{(R_i)} \widetilde{\pi} \right) \frac{(-t')}{\widetilde{\text{tr}}\underline{\chi}} (T\psi) (\widetilde{\text{tr}}\underline{\chi} (R_i^{\alpha'-1} T^l \underline{\Delta}\mu)) (R_i^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\tilde{g}} \\
& =: -III'_1 - III'_2 - III'_3 - III'_4.
\end{aligned} \tag{8.54}$$

In view of the pointwise estimates for $\text{tr}^{(R_i)} \widetilde{\pi}$ and $R_i \widetilde{\text{tr}}\underline{\chi}$, III'_2 and III'_4 are lower order with respect to δ, t and the order of derivative compared to III'_1 . Also the pointwise estimate for $R_i \widetilde{\text{tr}}\underline{\chi}$ implies that III'_3 is lower order with respect to the order of derivative compared to III'_1 . So here we only estimate III'_1 . Using Proposition 6.4, a preliminary estimate for III'_1 is given by

$$\begin{aligned}
|III'_1| & \lesssim \delta^{1/2} \int_{-r_0}^t \delta^l \|R_i^{\alpha'-1} T^l \underline{\Delta}\mu\|_{L^2(\Sigma_{t'}^{\underline{u}})} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\
& \lesssim \delta \int_{-r_0}^t \int_{-r_0}^{t'} (-t'')^{-2} \mu_m^{-b_{|\alpha|+2}}(t'') \left(\sqrt{\widetilde{E}_{\leq|\alpha|+2}(t'', \underline{u})} + \mu_m^{-1/2}(t'') (-t'')^{-1} \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t'', \underline{u})} \right) dt'' \\
& \quad \cdot \mu_m^{-b_{|\alpha|+2}-1/2}(t') (-t')^{-1} \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\
& \lesssim \delta \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}-1/2}(t') (-t')^{-1} \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \int_{-r_0}^{t'} (-t'')^{-2} dt'' dt' \\
& \quad + \delta \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}-1}(t') (-t')^{-1} \widetilde{E}_{\leq|\alpha|+2}(t', \underline{u}) \int_{-r_0}^{t'} (-t'')^{-3} dt'' dt'.
\end{aligned} \tag{8.55}$$

In the last step we used the second part of Lemma 7.1 and the monotonicity of $\widetilde{E}_{\leq|\alpha|+2}(t, \underline{u})$ and $\underline{E}_{\leq|\alpha|+2}(t, \underline{u})$. Splitting the time integral as “non-shock” and “shock” parts, III'_1 is bounded by

$$\begin{aligned}
& \delta \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}-1/2}(t') (-t')^{-2} \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \sqrt{\underline{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\
& + \delta \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}-1}(t') (-t')^{-2} \widetilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' \\
& \lesssim \delta \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^{t_0} (-t')^{-2} \left(\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u}) + \underline{E}_{\leq|\alpha|+2}(t', \underline{u}) \right) dt' \\
& + \frac{\delta \mu_m^{-2b_{|\alpha|+2}+1/2}(t)}{2b_{|\alpha|+2}-1/2} \widetilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \frac{\delta \mu_m^{-2b_{|\alpha|+2}}(t)}{2b_{|\alpha|+2}} \underline{E}_{\leq|\alpha|+2}(t, \underline{u}).
\end{aligned} \tag{8.56}$$

In view of (8.44) and the propagation equation for $\text{tr}\underline{\chi}$, I' is lower order with respect to t' compared to III' . Finally we consider the estimate for II' . Regarding to the contribution from the factor $(\underline{L} + \widetilde{\text{tr}}\underline{\chi}) R^{\alpha'} T^l \underline{\Delta}\mu$, we rewrite systematically

$$R_i^{\alpha'} T^l \Delta \mu = \Delta R_i^{\alpha'} T^l \mu + \mathcal{D}^2 R_i^{\alpha'-1} T^l \mu + \mathcal{D}^2 R_i^{\alpha'} T^{l-1} \mu + \mathcal{O}_{0,1}^{\leq 2} \left(\mathcal{D} R_i^{\alpha'-1} T^l \mu + \mathcal{D} R_i^{\alpha'} T^{l-1} \mu \right).$$

in view of (7.45). Here we abuse the notation by using \mathcal{D} and \mathcal{D}^2 to denote $\frac{\partial}{\partial \theta^A}$ and $\frac{\partial^2}{\partial \theta^A \partial \theta^B}$ respectively. Obviously the contribution of the last term above is lower order compared to the other three terms. While the contribution of the third term above is lower order with respect to δ compared to the first two terms. The contribution of the second term is

$$-\delta^{2l+2} \int_{W_{\underline{u}}^t} \frac{2(-t')}{\widetilde{\text{tr}} \chi} (T\psi) \left((\underline{L} + \widetilde{\text{tr}} \chi) (\mathcal{D}^2 R_i^{\alpha'-1} T^l \mu) \right) (R_i^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\widetilde{g}}. \quad (8.57)$$

In particular the contribution from the term involving “ $\widetilde{\text{tr}} \chi$ ” is lower order compared to III'_1 , which has been estimated. For the contribution from the term involving “ \underline{L} ”, the propagation equation for μ implies

$$\underline{L}(\mathcal{D}^2 R_i^{\alpha'-1} T^l \mu) = \mathcal{D}^2 R_i^{\alpha'-1} T^l m + \mathcal{D}^2 R_i^{\alpha'-1} T^l (\mu e) + {}^{(R_i)} \underline{Z} \mathcal{D} R_i^{\alpha'-1} T^l \mu + \Lambda \mathcal{D}^2 R_i^{\alpha'-1} T^{l-1} \mu. \quad (8.58)$$

The contributions of the last two terms are lower order compared to III'_1 and so are the terms in the second term for which e and its derivatives can be bounded in L^∞ . The contributions from the first term is bounded by

$$\begin{aligned} & \left| \delta^{2l+2} \int_{W_{\underline{u}}^t} \frac{2(-t')}{\widetilde{\text{tr}} \chi} (T\psi) (\mathcal{D}^2 R_i^{\alpha'-1} T^l m) (R_i^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\widetilde{g}} \right| \\ & \lesssim \delta \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\widetilde{E}_{\leq |\alpha|+2}(t', \underline{u})} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\widetilde{E}_{\leq |\alpha|+1}(t', \underline{u})} dt', \end{aligned} \quad (8.59)$$

which enjoys the same estimate as (8.56). The contribution from the terms in which the derivatives of e are bounded in L^2 is bounded by (Here we only consider the case that all the derivatives fall on e , which is the principal term.)

$$\begin{aligned} & \left| \delta^{2l+2} \int_{W_{\underline{u}}^t} \frac{2(-t')}{\widetilde{\text{tr}} \chi} (T\psi) (\mu \mathcal{D}^2 R_i^{\alpha'-1} T^l e) (R_i^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\widetilde{g}} \right| \\ & \lesssim \delta \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\widetilde{E}_{\leq |\alpha|+2}(t', \underline{u})} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\widetilde{E}_{\leq |\alpha|+1}(t', \underline{u})} dt', \end{aligned} \quad (8.60)$$

which also enjoys the same estimate as (8.56).

Finally we consider the contribution from the factor $(\underline{L} + \widetilde{\text{tr}} \chi) (\mathcal{D} R_i^{\alpha'} T^l \mu)$ to II' . Let us denote by $\widetilde{\text{div}}$ the divergence operator on $S_{t,\underline{u}}$ with respect to \widetilde{g} . A direct computation implies

$$(\underline{L} + \widetilde{\text{tr}} \chi) (\mathcal{D} R_i^{\alpha'} T^l \mu) = (\underline{L} + \widetilde{\text{tr}} \chi) \left(\widetilde{\text{div}} \left(\nabla_{\widetilde{g}} R_i^{\alpha'} T^l \mu \right) \right) = \widetilde{\text{div}} (\underline{L} + \widetilde{\text{tr}} \chi) \left(\nabla_{\widetilde{g}} R_i^{\alpha'} T^l \mu \right). \quad (8.61)$$

On the other hand since $\mathcal{D}_{\widetilde{g}} = c \mathcal{D}$, we have

$$(\underline{L} + \widetilde{\text{tr}} \chi) (\mathcal{D} R_i^{\alpha'} T^l \mu) = \underline{L} \left(\frac{1}{c} \right) c \mathcal{D} R_i^{\alpha'} T^l \mu + \frac{1}{c} (\underline{L} + \widetilde{\text{tr}} \chi) (\mathcal{D}_{\widetilde{g}} R_i^{\alpha'} T^l \mu). \quad (8.62)$$

The contribution of the first term on the right hand side above to II' is

$$-\delta^{2l+2} \int_{W_{\underline{u}}^t} \frac{2c(-t')}{\widetilde{\text{tr}}\underline{\chi}} \underline{L} \left(\frac{1}{c} \right) (T\psi) (\underline{\Delta} R_i^{\alpha'} T^l \mu) (R_i^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\widetilde{g}}, \quad (8.63)$$

which is lower order with respect to δ and t compared to III' . The contribution of the second term in (8.62) is bounded by

$$\begin{aligned} & -\delta^{2l+2} \int_{W_{\underline{u}}^t} \frac{2(-t')}{\widetilde{\text{ctr}}\underline{\chi}} (T\psi) \widetilde{\text{div}} \left((\underline{L} + \widetilde{\text{tr}}\underline{\chi}) \left(\nabla_{\widetilde{g}} R_i^{\alpha'} T^l \mu \right) \right) (R^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\widetilde{g}} \\ & = \delta^{2l+2} \int_{W_{\underline{u}}^t} \frac{2(-t')}{\widetilde{\text{ctr}}\underline{\chi}} (T\psi) \left((\underline{L} + \widetilde{\text{tr}}\underline{\chi}) \left(\nabla_{\widetilde{g}} R_i^{\alpha'} T^l \mu \right) \right) (\nabla_{\widetilde{g}} R^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\widetilde{g}} \\ & + \delta^{2l+2} \int_{W_{\underline{u}}^t} \nabla_{\widetilde{g}} \left(\frac{2(-t')}{\widetilde{\text{ctr}}\underline{\chi}} (T\psi) \right) \left((\underline{L} + \widetilde{\text{tr}}\underline{\chi}) \left(\nabla_{\widetilde{g}} R_i^{\alpha'} T^l \mu \right) \right) (R^{\alpha'} T^{l+1} \psi) dt' d\underline{u}' d\mu_{\widetilde{g}} \end{aligned} \quad (8.64)$$

In view of the fact $\nabla_{\widetilde{g}} = c\nabla$, the second term on the right hand side above is similar to (8.57) and therefore is also bounded as (8.56). The only difference between the first term on the right hand side above and (8.57) is that the variation $\nabla_{\widetilde{g}} R_i^{\alpha'} T^{l+1} \psi$ is top order, while the variation in (8.57) is one order less. Therefore similar to (8.59) and (8.60), this contribution is bounded as

$$\begin{aligned} & \delta \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt' \\ & + \delta \int_{-r_0}^t (-t')^{-2} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\widetilde{E}_{\leq|\alpha|+2}(t', \underline{u})} dt', \end{aligned} \quad (8.65)$$

which is finally bounded as (8.56).

9. TOP ORDER ENERGY ESTIMATES

Now we are ready to complete the top order energy estimates, namely, the energy estimates for the variations of order up to $|\alpha| + 2$. As we have pointed out, we allow the top order energies to blow up as shock forms. So in this section, we prove that the modified energies $\widetilde{E}_{\leq|\alpha|+2}(t, \underline{u})$, $\widetilde{E}_{\leq|\alpha|+2}(t, \underline{u})$ and $\widetilde{F}_{\leq|\alpha|+2}(t, \underline{u})$, $\widetilde{F}_{\leq|\alpha|+2}(t, \underline{u})$ are bounded by initial data. Therefore we obtain a rate for the possible blow up of the top order energies.

9.1. Estimates associated to K_1 . We start with the energy inequality for $Z_i^{\alpha'+1} \psi$ as we obtained in Section 6. Here Z_i is any one of R_i , Q and T .

$$\begin{aligned} & \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \left(\underline{E}[Z_i^{\alpha'+1} \psi](t, \underline{u}) + \underline{F}[Z_i^{\alpha'+1} \psi](t, \underline{u}) + K[Z_i^{\alpha'+1} \psi](t, \underline{u}) \right) \\ & \leq C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1} \psi](-r_0, \underline{u}) + C \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \int_{W_{\underline{u}}^t} c^{-2} \widetilde{Q}_{1, |\alpha'|+2} \end{aligned}$$

where l' is the number of T 's appearing in the string of $Z_i^{\alpha'}$. In the spacetime integral $\int_{W_{\underline{u}}^t} c^{-2} \widetilde{Q}_{1, |\alpha'|+2}$ we have the contributions from the deformation tensor of K_1 , which have been investigated in Section 6. Actually, if we choose N_{top} to be large enough, then we can bound $\|\not{d} Z_i^{\beta} \mu\|_{L^\infty(\Sigma_{\widetilde{t}}^{\underline{u}})}$ in terms of initial data by using the same argument as in Section 4.2 for $|\beta| \leq N_\infty + 1$.

Another contribution of the spacetime integral $\int_{W_{\underline{u}}^t} c^{-2} \tilde{Q}_{1,|\alpha'|+2}$ comes from $\int_{W_{\underline{u}}^t} \frac{1}{c} \tilde{\rho}_{|\alpha'|+2} \cdot \underline{L} Z_i^{\alpha'+1} \psi$, namely, the deformation tensor of commutators, which has been studied intensively in the last section. Among these we first consider the lower order optical contributions, which are bounded by (See (8.11) and (8.18)):

$$\begin{aligned}
& \delta^{1/2} K_{\leq|\alpha|+2}(t, \underline{u}) + \delta^{-1/2} \int_0^{\underline{u}} \underline{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \\
& + \int_{-r_0}^t (-t')^{-2} \underline{E}(t', \underline{u}) dt' + \delta \int_{-r_0}^t (-t')^{-2} E_{\leq|\alpha|+2}(t', \underline{u}) dt' \\
& \lesssim \mu_m^{-2b_{|\alpha|+2}}(t) \left(\delta^{1/2} \tilde{K}_{\leq|\alpha|+2}(t, \underline{u}) + \delta^{-1/2} \int_0^{\underline{u}} \tilde{\underline{F}}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \right. \\
& \left. + \int_{-r_0}^t \tilde{\underline{E}}_{\leq|\alpha|+2}(t', \underline{u}) dt' + \delta \int_{-r_0}^t (-t')^{-2} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' \right).
\end{aligned} \tag{9.1}$$

Here we define the following non-decreasing quantity in t :

$$\tilde{K}_{\leq|\alpha|+2}(t, \underline{u}) := \sup_{t' \in [-r_0, t]} \{ \mu_m^{2b_{|\alpha|+2}}(t') K_{\leq|\alpha|+2}(t', \underline{u}) \} \tag{9.2}$$

By (8.42), (8.43), (8.49), (8.52), (8.53), (8.56), the top order optical contributions associated to K_1 is bounded by (up to a constant)

$$\begin{aligned}
& \frac{1}{b_{|\alpha|+2} - 1/2} \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{\underline{E}}_{\leq|\alpha|+2}(t, \underline{u}) + \epsilon \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{\underline{E}}_{\leq|\alpha|+2}(t, \underline{u}) \\
& + C_\epsilon \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^t (-t')^{-2} \tilde{\underline{E}}_{\leq|\alpha|+2}(t', \underline{u}) dt' + \delta \mu_m^{-2b_{|\alpha|+2}}(t) \tilde{\underline{E}}_{\leq|\alpha|+2}(t, \underline{u}) \\
& + \delta \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^t (-t')^{-2} \left(\tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) + \underline{\tilde{E}}_{\leq|\alpha|+2}(t', \underline{u}) \right) dt' \\
& + \frac{\delta \mu_m^{-2b_{|\alpha|+2}+1/2}(t)}{b_{|\alpha|+2} - 1} \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} \sqrt{\underline{\tilde{E}}_{\leq|\alpha|+2}(t, \underline{u})} \\
& + \frac{\delta \mu_m^{-2b_{|\alpha|+2}+1/2}(t)}{2b_{|\alpha|+2} - 1/2} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}).
\end{aligned} \tag{9.3}$$

Substituting these contributions into the energy inequality, and use the fact that $\mu_m(t) \leq 1$, we obtain:

$$\begin{aligned}
& \mu_m^{2b_{|\alpha|+2}}(t) \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \left(\underline{E}[Z_i^{\alpha'+1}\psi](t, \underline{u}) + \underline{F}[Z_i^{\alpha'+1}\psi](t, \underline{u}) + K[Z_i^{\alpha'+1}\psi](t, \underline{u}) \right) \\
& \lesssim \mu_m^{2b_{|\alpha|+2}}(t) \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + \frac{1}{b_{|\alpha|+2} - 1/2} \tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u}) \\
& \quad + \epsilon \tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u}) + C_\epsilon \int_{-r_0}^t (-t')^{-2} \tilde{\underline{E}}_{\leq |\alpha|+2}(t', \underline{u}) dt' + \delta \tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u}) \\
& \quad + \delta \int_{-r_0}^t (-t')^{-2} \left(\tilde{\underline{E}}_{\leq |\alpha|+2}(t', \underline{u}) + \underline{\tilde{E}}_{\leq |\alpha|+2}(t', \underline{u}) \right) dt' \\
& \quad + \frac{\delta}{b_{|\alpha|+2} - 1} \sqrt{\tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u})} \sqrt{\underline{\tilde{E}}_{\leq |\alpha|+2}(t, \underline{u})} + \frac{\delta}{2b_{|\alpha|+2} - 1/2} \tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u}).
\end{aligned}$$

Now the right hand side of the above inequality is non-decreasing in t , so the above inequality is also valid if we replace “ t ” by any $t' \in [-r_0, t]$ on the left hand side:

$$\begin{aligned}
& \mu_m^{2b_{|\alpha|+2}}(t') \sum_{|\alpha'| \leq |\alpha|} \left(\underline{E}[Z_i^{\alpha'+1}\psi](t', \underline{u}) + \underline{F}[Z_i^{\alpha'+1}\psi](t', \underline{u}) + K[Z_i^{\alpha'+1}\psi](t', \underline{u}) \right) \\
& \lesssim \mu_m^{2b_{|\alpha|+2}}(t') \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + \frac{1}{b_{|\alpha|+2} - 1/2} \tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u}) \\
& \quad + \epsilon \tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u}) + C_\epsilon \int_{-r_0}^t (-t')^{-2} \tilde{\underline{E}}_{\leq |\alpha|+2}(t', \underline{u}) dt' + \delta \tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u}) \\
& \quad + \delta \int_{-r_0}^t (-t')^{-2} \left(\tilde{\underline{E}}_{\leq |\alpha|+2}(t', \underline{u}) + \underline{\tilde{E}}_{\leq |\alpha|+2}(t', \underline{u}) \right) dt' \\
& \quad + \frac{\delta}{b_{|\alpha|+2} - 1} \sqrt{\tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u})} \sqrt{\underline{\tilde{E}}_{\leq |\alpha|+2}(t, \underline{u})} + \frac{\delta}{2b_{|\alpha|+2} - 1/2} \tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u}).
\end{aligned}$$

For each term in the sum on the left hand side of above inequality, we keep it on the left hand side and ignore all the other terms. Then taking supremum of the term we kept with respect to $t' \in [-r_0, t]$. Repeat this process for all the terms on the left hand side, we finally obtain:

$$\begin{aligned}
& \tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u}) + \underline{\tilde{F}}_{\leq |\alpha|+2}(t, \underline{u}) + \tilde{K}_{\leq |\alpha|+2}(t, \underline{u}) \\
& \lesssim \mu_m^{2b_{|\alpha|+2}}(t) \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + \boxed{\frac{1}{b_{|\alpha|+2} - 1/2} \tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u})} \\
& \quad + \epsilon \tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u}) + C_\epsilon \int_{-r_0}^t (-t')^{-2} \tilde{\underline{E}}_{\leq |\alpha|+2}(t', \underline{u}) dt' + \delta \tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u}) \\
& \quad + \delta \int_{-r_0}^t (-t')^{-2} \left(\tilde{\underline{E}}_{\leq |\alpha|+2}(t', \underline{u}) + \underline{\tilde{E}}_{\leq |\alpha|+2}(t', \underline{u}) \right) dt' \\
& \quad + \frac{\delta}{b_{|\alpha|+2} - 1} \sqrt{\tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u})} \sqrt{\underline{\tilde{E}}_{\leq |\alpha|+2}(t, \underline{u})} + \frac{\delta}{2b_{|\alpha|+2} - 1/2} \tilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u}).
\end{aligned}$$

The control on the boxed term relies on Remark 8.2—since $\frac{C}{b_{|\alpha|+2}}$ is suitably small, the boxed term can be absorbed by the left hand side. So if δ and ϵ are appropriately small, we obtain

$$\begin{aligned}
& \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{K}_{\leq|\alpha|+2}(t, \underline{u}) \\
& \lesssim \mu_m^{2b_{|\alpha|+2}}(t) \sum_{|\alpha'|\leq|\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + \frac{\delta}{b_{|\alpha|+2}-1} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \\
& + \int_{-r_0}^t (-t')^{-2} \left(\delta \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) + \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) \right) dt',
\end{aligned} \tag{9.4}$$

which implies, by Gronwall,

$$\begin{aligned}
& \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{K}_{\leq|\alpha|+2}(t, \underline{u}) \\
& \lesssim \mu_m^{2b_{|\alpha|+2}}(t) \sum_{|\alpha'|\leq|\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) \\
& + \frac{\delta}{b_{|\alpha|+2}-1} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \int_{-r_0}^t (-t')^{-2} \delta \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt'.
\end{aligned} \tag{9.5}$$

9.2. Estimates associated to K_0 . Now we turn to the top order energy estimates for K_0 . We start with the energy identity for $Z_i^{\alpha'+1}\psi$, where Z_i is any one of R_i , Q and T :

$$\begin{aligned}
& \sum_{|\alpha'|\leq|\alpha|} \delta^{2l'} \left(E[Z_i^{\alpha'+1}\psi](t, \underline{u}) + F[Z_i^{\alpha'+1}\psi](t, \underline{u}) \right) \\
& \lesssim \sum_{|\alpha'|\leq|\alpha|} \delta^{2l'} E[Z_i^{\alpha'+1}\psi](-r_0, \underline{u}) + \sum_{|\alpha'|\leq|\alpha|} \int_{W_{\underline{u}}^t} c^{-2} \tilde{Q}_{0,|\alpha'|+2}
\end{aligned}$$

Again, l' is the number of T s' in the string of $Z_i^{\alpha'+1}$.

In the spacetime integral $\int_{W_{\underline{u}}^t} c^{-2} \tilde{Q}_{0,|\alpha'|+2}$ we have the contributions from the deformation tensor of K_0 , which have been investigated in section 6 and also the contribution of the spacetime integral from $\int_{W_{\underline{u}}^t} \frac{1}{c} \tilde{\rho}_{|\alpha'|+2} \cdot LZ_i^{\alpha'+1}\psi$, namely, the deformation tensor of commutators, which has been studied intensively in the last section. We first consider the lower order optical contributions, which are bounded by (See (8.12), (8.19) and (9.5) and provided that δ is sufficiently small):

$$\begin{aligned}
& \mu_m^{-2b_{|\alpha|+2}}(t) \left(\int_{-r_0}^t (-t')^{-3/2} \delta^{1/2} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' + \delta^{1/2} \tilde{K}_{\leq|\alpha|+2}(t, \underline{u}) + \delta^{-1/2} \int_0^{\underline{u}} \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}') d\underline{u}' \right) \\
& \lesssim \underline{E}_{\leq|\alpha|+2}(-r_0, \underline{u}) + \mu_m^{-2b_{|\alpha|+2}}(t) \left(\int_{-r_0}^t (-t')^{-3/2} \delta^{1/2} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt' + \frac{\delta^{3/2}}{b_{|\alpha|+2}-1} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) \right).
\end{aligned} \tag{9.6}$$

Here we used the following fact: Since the right hand side of (9.5) is non-decreasing in \underline{u} , $\sup_{\underline{u}' \in [0, \underline{u}]} \{ \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}') \}$ is also bounded by the right hand side of (9.5).

By (8.23), (8.25), (8.29), (8.30) and (9.5), the top order optical contributions are bounded by (up to a constant)

$$\frac{\mu_m^{-2b_{|\alpha|+2}}(t)}{2b_{|\alpha|+2}} \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \mu_m^{-2b_{|\alpha|+2}}(t) \int_{-r_0}^t (-t')^{-2} \tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) dt'. \tag{9.7}$$

Substituting (9.6) and (9.7) into the energy inequality we obtain

$$\begin{aligned}
& \mu_m^{2b_{|\alpha|+2}}(t)E_{\leq|\alpha|+2}(t, \underline{u}) + \mu_m^{2b_{|\alpha|+2}}(t)F_{\leq|\alpha|+2}(t, \underline{u}) \\
& \lesssim E_{\leq|\alpha|+2}(-r_0, \underline{u}) + \underline{E}_{\leq|\alpha|+2}(-r_0, \underline{u}) \\
& \quad + \frac{1}{2b_{|\alpha|+2}}\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \int_{-r_0}^t (-t')^{-3/2}\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})dt',
\end{aligned} \tag{9.8}$$

which, using the fact that the right hand side above is non-decreasing in t , implies

$$\begin{aligned}
& \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}) \\
& \lesssim E_{\leq|\alpha|+2}(-r_0, \underline{u}) + \underline{E}_{\leq|\alpha|+2}(-r_0, \underline{u}) \\
& \quad + \frac{1}{2b_{|\alpha|+2}}\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \int_{-r_0}^t (-t')^{-3/2}\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})dt'.
\end{aligned} \tag{9.9}$$

Using Gronwall and taking $b_{|\alpha|+2}$ large enough implies

$$\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}) \lesssim E_{\leq|\alpha|+2}(-r_0, \underline{u}) + \underline{E}_{\leq|\alpha|+2}(-r_0, \underline{u}). \tag{9.10}$$

Substituting this into (9.5) we obtain

$$\begin{aligned}
& \tilde{\underline{E}}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{\underline{F}}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{\underline{K}}_{\leq|\alpha|+2}(t, \underline{u}) \\
& \lesssim \underline{E}_{\leq|\alpha|+2}(-r_0, \underline{u}) + \delta E_{\leq|\alpha|+2}(-r_0, \underline{u}).
\end{aligned} \tag{9.11}$$

This completes the top order energy estimates. If we denote the initial energies by

$$\mathcal{D}_{|\alpha|+2}^u := E_{\leq|\alpha|+2}(-r_0, \underline{u}) + \delta^{-1}\underline{E}_{\leq|\alpha|+2}(-r_0, \underline{u}), \tag{9.12}$$

then the top order energy estimates can be summarized as

$$\begin{aligned}
& \tilde{\underline{E}}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{\underline{F}}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{\underline{K}}_{\leq|\alpha|+2}(t, \underline{u}) \lesssim \delta \mathcal{D}_{|\alpha|+2}^u \\
& \tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \tilde{F}_{\leq|\alpha|+2}(t, \underline{u}) \lesssim \mathcal{D}_{|\alpha|+2}^u.
\end{aligned} \tag{9.13}$$

10. DESCENT SCHEME

In the previous section, we have shown that the modified energies $\tilde{E}_{\leq|\alpha|+2}(t), \tilde{\underline{E}}_{\leq|\alpha|+2}(t)$ for the top order variations are bounded by the initial energies $\mathcal{D}_{|\alpha|+2}^u$. According to the definition, the modified energies go to zero when $\mu_m(t)$ goes to zero. This means the energy estimates obtained in the last section are not sufficient for us to close the argument when shock forms. However, based on those estimates, we shall show in this section, that if the order of derivative decreases, the power of $\mu_m(t)$ needed in the definition of modified energies also decreases. The key point is that after several steps, this power could be zero and the energies which do not go to zero as shock forms can be bounded.

10.1. Next-to-top order error estimates. We first investigate the estimates associated to K_1 . To improve the energy estimates for the next-to-the-top variations, we consider the spacetime integral (Keep in mind that

the top order quantities are of order $|\alpha| + 2$):

$$\begin{aligned}
& \left| \int_{W_{\underline{u}}^t} \frac{1}{c} \frac{2(-t)}{\widetilde{\text{tr}} \underline{\chi}} (T\psi) \cdot (Z_i^\alpha \text{tr} \underline{\chi}') \cdot \left(\underline{L} Z_i^\alpha \psi + \frac{1}{2} \widetilde{\text{tr}} \underline{\chi} Z_i^\alpha \psi \right) dt' d\underline{u}' d\mu_{\tilde{g}} \right| \\
& \lesssim \delta^{-1/2} \int_{W_{\underline{u}}^t} (-t') |Z_i^\alpha \text{tr} \underline{\chi}'| \left| \underline{L} Z_i^\alpha \psi + \frac{1}{2} \widetilde{\text{tr}} \underline{\chi} Z_i^\alpha \psi \right| dt' d\underline{u}' d\mu_{\tilde{g}} \\
& \lesssim \delta^{-1/2} \left(\int_{W_{\underline{u}}^t} |Z_i^\alpha \text{tr} \underline{\chi}'|^2 dt' d\underline{u}' d\mu_{\tilde{g}} \right)^{1/2} \cdot \left(\int_{W_{\underline{u}}^t} (-t')^2 \left(\underline{L} Z_i^\alpha \psi + \frac{1}{2} \widetilde{\text{tr}} \underline{\chi} \psi \right)^2 dt' d\underline{u}' d\mu_{\tilde{g}} \right)^{1/2} \\
& \lesssim \delta^{-1/2} \left(\int_{-r_0}^t \|Z_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})}^2 dt' \right)^{1/2} \cdot \left(\int_0^{\underline{u}} \underline{F}[Z_i^\alpha \psi](t, \underline{u}') d\underline{u}' \right)^{1/2}
\end{aligned} \tag{10.1}$$

Throughout this subsection, Z_i is either R_i or Q .

By Proposition 6.3,

$$\begin{aligned}
\|Z_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})} & \lesssim \delta^{1/2} \int_{-r_0}^t (-t')^{-3} \mu_m^{-1/2}(t') \sqrt{\underline{E}_{|\alpha|+2}(t', \underline{u})} dt' \\
& \lesssim \delta^{1/2} \int_{-r_0}^t (-t')^{-3} \mu_m^{-1/2-b_{|\alpha|+2}}(t') \sqrt{\widetilde{\underline{E}}_{\leq |\alpha|+2}(t', \underline{u})} dt' \\
& \lesssim \delta^{1/2} \sqrt{\widetilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u})} \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') (-t')^{-3} dt' \\
& \lesssim \delta^{1/2} (-t)^{-1} \sqrt{\widetilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u})} \int_{-r_0}^t \mu_m^{-b_{|\alpha|+2}-1/2}(t') (-t')^{-2} dt' \\
& \lesssim \delta^{1/2} \mu_m^{-b_{|\alpha|+2}+1/2}(t) (-t)^{-1} \sqrt{\widetilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u})}.
\end{aligned} \tag{10.2}$$

Then by the top order energy estimates obtained in the last section, the integral in the first factor of (10.1) is bounded by (up to a constant):

$$\begin{aligned}
& \delta \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}+1}(t') (-t')^{-2} \widetilde{\underline{E}}_{\leq |\alpha|+2}(t', \underline{u}) dt' \\
& \lesssim \delta \widetilde{\underline{E}}_{\leq |\alpha|+2}(t, \underline{u}) \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}+1}(t') (-t')^{-2} dt' \\
& \lesssim \delta^2 \mu_m^{-2b_{|\alpha|+2}+2}(t) \mathcal{D}_{|\alpha|+2}^{\underline{u}}
\end{aligned}$$

On the other hand, the second factor in (10.1) is bounded by:

$$\int_0^{\underline{u}} \underline{F}[Z_i^\alpha \psi](t, \underline{u}') d\underline{u}' \leq \mu_m^{-2b_{|\alpha|+1}}(t) \int_0^{\underline{u}} \sup_{t' \in [-r_0, t]} \{ \mu_m^{2b_{|\alpha|+1}}(t') \underline{F}[Z_i^\alpha \psi](t', \underline{u}') \} d\underline{u}'$$

where $b_{|\alpha|+1} = b_{|\alpha|+2} - 1$. Therefore (10.1) is bounded by (up to a constant):

$$\begin{aligned}
& \delta^{1/2} \mu_m^{-2b_{|\alpha|+1}}(t) \sqrt{\mathcal{D}_{|\alpha|+2}^{\underline{u}}} \sqrt{\int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}'} \\
& \lesssim \delta^2 \mu_m^{-2b_{|\alpha|+1}}(t) \mathcal{D}_{|\alpha|+2}^{\underline{u}} + C \delta^{-1} \mu_m^{-2b_{|\alpha|+1}}(t) \int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}'.
\end{aligned} \tag{10.3}$$

Next we consider the spacetime integral

$$\begin{aligned}
& \delta^{2l'+2} \left| \int_{W_{\underline{u}}^t} \frac{1}{c} \frac{2(-t')}{\tilde{\text{tr}} \underline{\chi}} (T\psi) \cdot (Z_i^{\alpha'} T^{l'} \Delta \mu) \cdot \left(\underline{L} Z_i^{\alpha'} T^{l'+1} \psi + \frac{1}{2} \tilde{\text{tr}} \underline{\chi} Z_i^{\alpha'} T^{l'+1} \psi \right) dt' d\underline{u}' d\mu_{\tilde{g}} \right| \\
& \lesssim \delta^{-1/2} \left(\int_{-r_0}^t \delta^{l'+1} \|Z_i^{\alpha'} T^{l'} \Delta \mu\|_{L^2(\Sigma_{t'}^{\underline{u}})}^2 dt' \right)^{1/2} \left(\int_0^{\underline{u}} \delta^{l'+1} \underline{F}[Z_i^{\alpha'} T^{l'+1} \psi](t, \underline{u}') d\underline{u}' \right)^{1/2}
\end{aligned} \tag{10.4}$$

with $|\alpha'| + l' \leq |\alpha| - 1$. By Proposition 6.4,

$$\begin{aligned}
& \delta^{l'+1} \|Z_i^{\alpha'} T^{l'} \Delta \mu\|_{L^2(\Sigma_t^{\underline{u}})} \\
& \lesssim \delta^{1/2} (-t)^{-1} \int_{-r_0}^t (-t')^{-2} \left(\sqrt{E_{\leq|\alpha|+2}(t', \underline{u})} + \mu_m^{-1/2}(t') (-t')^{-1} \sqrt{\underline{E}_{\leq|\alpha|+2}(t', \underline{u})} \right) dt' \\
& \lesssim \delta^{1/2} (-t)^{-1} \int_{-r_0}^t (-t')^{-2} \left(\mu_m^{-b_{|\alpha|+2}}(t') \sqrt{\tilde{E}_{\leq|\alpha|+2}(t', \underline{u})} + \mu_m^{-b_{|\alpha|+2}-1/2}(t') \sqrt{\underline{\tilde{E}}_{\leq|\alpha|+2}(t', \underline{u})} \right) dt' \\
& \lesssim \delta^{1/2} (-t)^{-1} \left(\mu_m^{-b_{|\alpha|+2}+1/2}(t) \sqrt{\tilde{E}_{\leq|\alpha|+2}(t, \underline{u})} + \mu_m^{-b_{|\alpha|+2}+1}(t) \sqrt{\underline{\tilde{E}}_{\leq|\alpha|+2}(t, \underline{u})} \right).
\end{aligned} \tag{10.5}$$

Then by the top order energy estimates obtained in the last section, the integral in the first factor of (10.4) is bounded by $(\mu_m(t) \leq 1)$:

$$\begin{aligned}
& \delta \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}+1}(t') (-t')^{-2} \left(\tilde{E}_{\leq|\alpha|+2}(t', \underline{u}) + \underline{\tilde{E}}_{\leq|\alpha|+2}(t', \underline{u}) \right) dt' \\
& \lesssim \delta \left(\tilde{E}_{\leq|\alpha|+2}(t) + \underline{\tilde{E}}_{\leq|\alpha|+2}(t, \underline{u}) \right) \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2}+1}(t') (-t')^{-2} dt' \\
& \lesssim \delta \mu_m^{-2b_{|\alpha|+2}+2}(t) \left(\tilde{E}_{\leq|\alpha|+2}(t, \underline{u}) + \underline{\tilde{E}}_{\leq|\alpha|+2}(t, \underline{u}) \right) \\
& \lesssim \delta \mu_m^{-2b_{|\alpha|+2}+2}(t) \mathcal{D}_{|\alpha|+2}^{\underline{u}}.
\end{aligned}$$

Again, with $b_{|\alpha|+1} = b_{|\alpha|+2} - 1$, the spacetime integral (10.4) is bounded by (up to a constant):

$$\begin{aligned}
& \mu_m^{-2b_{|\alpha|+1}}(t) \sqrt{\mathcal{D}_{|\alpha|+2}^{\underline{u}}} \sqrt{\int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}'} \\
& \lesssim \delta \mu_m^{-2b_{|\alpha|+1}}(t) \mathcal{D}_{|\alpha|+2}^{\underline{u}} + C \delta^{-1} \mu_m^{-2b_{|\alpha|+1}}(t) \int_0^{\underline{u}} \tilde{E}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}'.
\end{aligned}$$

We proceed to consider the spacetime error integral associated to K_0 . We first consider the spacetime integral:

$$\begin{aligned} & \int_{W_{\underline{u}}^t} |T\psi| |Z_i^\alpha \text{tr} \underline{\chi}'| |LZ_i^\alpha \psi| dt' d\underline{u}' d\mu_{\tilde{g}} \\ & \lesssim \delta^{-1/2} \int_{-r_0}^t (-t')^{-1} \|Z_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})} \|LZ_i^\alpha \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \end{aligned}$$

Substituting the estimates:

$$\begin{aligned} \|Z_i^\alpha \text{tr} \underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})} & \lesssim \delta^{1/2} \mu_m^{-b_{|\alpha|+2}+1/2}(t') (-t')^{-1} \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} \\ & \lesssim \delta \mu_m^{-b_{|\alpha|+2}+1/2}(t) (-t')^{-1} \sqrt{\mathcal{D}_{|\alpha|+2}^{\underline{u}}}, \\ \|LZ_i^\alpha \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} & \lesssim \mu_m^{-b_{|\alpha|+1}}(t') \sqrt{\tilde{E}_{\leq |\alpha|+1}(t', \underline{u})} \end{aligned}$$

with $b_{|\alpha|+1} = b_{|\alpha|+2} - 1$, and using the fact that $\tilde{E}_{\leq |\alpha|+1}(t)$ are non-decreasing in t , we see that the spacetime integral is bounded by ($\mu_m(t) \leq 1$):

$$\begin{aligned} & \delta^{1/2} \sqrt{\mathcal{D}_{|\alpha|+2}^{\underline{u}}} \sqrt{\tilde{E}_{\leq |\alpha|+1}(t, \underline{u})} \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+1}-1/2}(t') (-t')^{-2} dt' \\ & \lesssim \delta^{1/2} \mu_m^{-2b_{|\alpha|+1}+1/2}(t) \sqrt{\mathcal{D}_{|\alpha|+2}^{\underline{u}}} \sqrt{\tilde{E}_{\leq |\alpha|+1}(t, \underline{u})} \\ & \lesssim \mu_m^{-2b_{|\alpha|+1}}(t) \mathcal{D}_{|\alpha|+2}^{\underline{u}} + \delta \mu_m^{-2b_{|\alpha|+1}}(t) \tilde{E}_{\leq |\alpha|+1}(t, \underline{u}) \end{aligned} \tag{10.6}$$

Finally, we consider the spacetime integral:

$$\begin{aligned} & \delta^{2l'+2} \int_{W_{\underline{u}}^t} |Z_i^{\alpha'} T^{l'} \Delta \mu| |T\psi| |LZ_i^{\alpha'} T^{l'+1} \psi| dt' d\underline{u}' d\mu_{\tilde{g}} \\ & \lesssim \delta^{2l'+2-1/2} \int_{-r_0}^t (-t')^{-1} \|Z_i^{\alpha'} T^{l'} \Delta \mu\|_{L^2(\Sigma_{t'}^{\underline{u}})} \|LZ_i^{\alpha'} T^{l'+1} \psi\|_{L^2(\Sigma_{t'}^{\underline{u}})} dt' \end{aligned} \tag{10.7}$$

for $|\alpha'| + l' \leq |\alpha| - 1$. Again, substituting the estimates ($\mu_m(t) \leq 1$):

$$\delta^{l'+1} \|Z_i^{\alpha'} T^{l'} \Delta \mu\|_{L^2(\Sigma_{t'}^{\underline{u}})} \lesssim \delta^{1/2} \mu_m^{-b_{|\alpha|+2}+1/2}(t') (-t')^{-1} \left(\sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} + \sqrt{\tilde{E}_{\leq |\alpha|+2}(t', \underline{u})} \right)$$

with $b_{|\alpha|+1} = b_{|\alpha|+2} - 1$, the same argument implies that the spacetime integral is bounded by:

$$\begin{aligned} & \sqrt{\mathcal{D}_{|\alpha|+2}^{\underline{u}}} \sqrt{\tilde{E}_{\leq |\alpha|+1}(t, \underline{u})} \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+1}-1/2}(t') (-t')^{-2} dt' \\ & \lesssim \mu_m^{-2b_{|\alpha|+1}+1/2}(t) \sqrt{\mathcal{D}_{|\alpha|+2, l'}^{\underline{u}}} \sqrt{\tilde{E}_{\leq |\alpha|+1}(t, \underline{u})} \\ & \leq C_\epsilon \mu_m^{-2b_{|\alpha|+1}}(t) \mathcal{D}_{|\alpha|+2}^{\underline{u}} + \epsilon \tilde{E}_{\leq |\alpha|+1}(t, \underline{u}). \end{aligned} \tag{10.8}$$

Here ϵ is a small absolute positive constant.

10.2. Energy estimates-next to top order. Throughout this subsection Z_i could be R_i, Q and T . Now we consider the other contributions from the spacetime error integrals associated to K_1 . For the variations $Z_i^{\alpha'} \psi$ where $|\alpha'| \leq |\alpha|$, the contributions similar to (8.18) are bounded by (up to a constant)

$$\begin{aligned}
& \delta^{1/2} K_{\leq|\alpha|+1}(t, \underline{u}) + \delta^{-1/2} \int_0^{\underline{u}} \underline{F}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}' \\
& + \int_{-r_0}^t (-t')^{-2} \underline{E}_{\leq|\alpha|+1}(t', \underline{u}) dt' + \delta \int_{-r_0}^t (-t')^{-2} E_{\leq|\alpha|+1}(t', \underline{u}) dt' \\
& \lesssim \delta^{1/2} \mu_m^{-2b_{|\alpha|+1}}(t) \tilde{K}_{\leq|\alpha|+1}(t, \underline{u}) + \delta^{-1/2} \mu_m^{-2b_{|\alpha|+1}}(t) \int_0^{\underline{u}} \tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}' \\
& + \mu_m^{-2b_{|\alpha|+1}}(t) \int_{-r_0}^t (-t')^{-2} \tilde{\underline{E}}_{\leq|\alpha|+1}(t', \underline{u}) dt' + \delta \mu_m^{-2b_{|\alpha|+1}}(t) \int_{-r_0}^t (-t')^{-2} \tilde{E}_{\leq|\alpha|+1}(t', \underline{u}) dt'.
\end{aligned} \tag{10.9}$$

In view of (10.3), (10.6), (10.9) and multiplying $\mu_m^{2b_{|\alpha|+1}}(t)$ on both sides of the energy inequality associated to K_1 for $Z_i^{\alpha'} \psi$ with $|\alpha'| \leq |\alpha|$ gives us

$$\begin{aligned}
& \sum_{|\alpha'| \leq |\alpha|} \mu_m^{2b_{|\alpha|+1}}(t) \delta^{2l'} \left(\underline{E}[Z_i^{\alpha'} \psi](t, \underline{u}) + \underline{F}[Z_i^{\alpha'} \psi](t, \underline{u}) + K[Z_i^{\alpha'} \psi](t, \underline{u}) \right) \\
& \lesssim \sum_{|\alpha'| \leq |\alpha|} \delta^{2l'} \underline{E}[Z_i^{\alpha'} \psi](-r_0, \underline{u}) + \delta^{-1} \int_0^{\underline{u}} \tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}' + \int_{-r_0}^t (-t')^{-2} \tilde{\underline{E}}(t', \underline{u}) dt' \\
& + \delta \int_{-r_0}^t (-t')^{-2} \tilde{\underline{E}}_{\leq|\alpha|+1}(t', \underline{u}) dt' + \delta^{1/2} \tilde{K}_{\leq|\alpha|+1}(t, \underline{u}) + \delta \mathcal{D}_{|\alpha|+2}^{\underline{u}}
\end{aligned}$$

Arguing as in the previous section, this inequality holds t is replaced by $t' \in [-r_0, t]$ on the left hand side. Taking the supremum with respect to $t' \in [-r_0, t]$ we obtain

$$\begin{aligned}
& \tilde{\underline{E}}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{K}_{\leq|\alpha|+1}(t, \underline{u}) \\
& \lesssim \delta \mathcal{D}_{|\alpha|+2}^{\underline{u}} + \delta^{-1} \int_0^{\underline{u}} \tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}' + \int_{-r_0}^t (-t')^{-2} \tilde{\underline{E}}_{\leq|\alpha|+2}(t', \underline{u}) dt' \\
& + \delta \int_{-r_0}^t (-t')^{-2} \tilde{\underline{E}}_{\leq|\alpha|+1}(t', \underline{u}) dt' + \delta^{1/2} \tilde{K}_{\leq|\alpha|+1}(t, \underline{u}).
\end{aligned}$$

Choosing δ sufficiently small and using Gronwall implies

$$\tilde{\underline{E}}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u}) + \tilde{K}_{\leq|\alpha|+1}(t, \underline{u}) \lesssim \delta \mathcal{D}_{|\alpha|+2}^{\underline{u}} + \delta \int_{-r_0}^t (-t')^{-2} \tilde{\underline{E}}_{\leq|\alpha|+1}(t', \underline{u}) dt'. \tag{10.10}$$

Next we consider the energy estimates associated to K_0 . We start with the variation $Z_i^{\alpha'} \psi$ with $|\alpha'| \leq |\alpha|$. The contributions similar to (8.19) is bounded by (up to a constant)

$$\begin{aligned}
& \delta^{-1/2} \int_0^{\underline{u}} \underline{F}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}' + \delta^{1/2} \int_{-r_0}^t (-t')^{-2} E_{\leq|\alpha|+1}(t', \underline{u}) dt' + \delta^{1/2} K_{\leq|\alpha|+1}(t, \underline{u}) \\
& \lesssim \delta^{-1/2} \mu_m^{-2b_{|\alpha|+1}}(t) \int_0^{\underline{u}} \tilde{\underline{F}}_{\leq|\alpha|+1}(t, \underline{u}') d\underline{u}' + \delta^{1/2} \mu_m^{-2b_{|\alpha|+1}}(t) \int_{-r_0}^t (-t')^{-2} \tilde{\underline{E}}_{\leq|\alpha|+1}(t', \underline{u}) dt' \\
& + \delta^{1/2} \mu_m^{-2b_{|\alpha|+1}}(t) \tilde{K}_{\leq|\alpha|+1}(t, \underline{u}) \\
& \lesssim \delta^{3/2} \mu_m^{-2b_{|\alpha|+1}}(t) \mathcal{D}_{|\alpha|+2}^{\underline{u}} + \delta^{3/2} \mu_m^{-2b_{|\alpha|+1}}(t) \int_{-r_0}^t (-t')^{-2} \tilde{\underline{E}}_{\leq|\alpha|+1}(t', \underline{u}) dt'.
\end{aligned} \tag{10.11}$$

Here in the last step we used (10.10). (10.11) together with (10.8) gives the following estimate

$$\begin{aligned} & \sum_{|\alpha'| \leq |\alpha|} \mu_m^{2b_{|\alpha|+1}}(t) \delta^{2l'} \left(E[Z_i^{\alpha'} \psi](t, \underline{u}) + F[Z_i^{\alpha'} \psi](t, \underline{u}) \right) \\ & \lesssim \mathcal{D}_{|\alpha|+2}^{\underline{u}} + \epsilon \tilde{E}_{\leq |\alpha|+1}(t, \underline{u}) + \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \tilde{E}_{\leq |\alpha|+1}(t', \underline{u}) dt', \end{aligned}$$

which implies

$$\begin{aligned} & \tilde{E}_{\leq |\alpha|+1}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+1}(t, \underline{u}) \\ & \lesssim \mathcal{D}_{|\alpha|+2}^{\underline{u}} + \epsilon \tilde{E}_{\leq |\alpha|+1}(t, \underline{u}) + \delta^{1/2} \int_{-r_0}^t (-t')^{-2} \tilde{E}_{\leq |\alpha|+1}(t', \underline{u}) dt' \end{aligned}$$

Choosing ϵ sufficiently small and using Gronwall, we finally have:

$$\tilde{E}_{\leq |\alpha|+1}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+1}(t, \underline{u}) \lesssim \mathcal{D}_{|\alpha|+2}^{\underline{u}} \quad (10.12)$$

Now substituting (10.12) to the right hand side of (10.10), we have:

$$\tilde{E}_{\leq |\alpha|+1}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+1}(t, \underline{u}) + \tilde{K}_{\leq |\alpha|+1}(t, \underline{u}) \lesssim \delta \mathcal{D}_{|\alpha|+2}^{\underline{u}}$$

Summarizing, we have:

$$\begin{aligned} & \tilde{E}_{\leq |\alpha|+1}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+1}(t, \underline{u}) + \tilde{K}_{\leq |\alpha|+1}(t, \underline{u}) \lesssim \delta \mathcal{D}_{|\alpha|+2}^{\underline{u}} \\ & \tilde{E}_{\leq |\alpha|+1}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+1}(t, \underline{u}) \lesssim \mathcal{D}_{|\alpha|+2}^{\underline{u}}. \end{aligned} \quad (10.13)$$

10.3. Descent scheme. We proceed in this way taking at the n th step:

$$b_{|\alpha|+2-n} = b_{|\alpha|+2} - n, \quad b_{|\alpha|+1-n} = b_{|\alpha|+2} - n - 1$$

in the role of $b_{|\alpha|+2}$ and $b_{|\alpha|+1}$ respectively, the argument beginning in the paragraph containing (10.1) and concluding with (10.13) being step 0. The n th step is exactly the same as the 0th step as above, as long as $b_{|\alpha|+1-n} > 0$. If we choose

$$b_{|\alpha|+2} = [b_{|\alpha|+2}] + \frac{3}{4} \quad (10.14)$$

where $[b_{|\alpha|+2}]$ is the integer part of $b_{|\alpha|+2}$, then $b_{|\alpha|+1-n} > 0$ is equivalent to $n \leq [b_{|\alpha|+2}] - 1$. For each of such n , we need to estimate the integrals:

$$\int_{-r_0}^t \mu_m^{-b_{|\alpha|+2-n}-1/2}(t') (-t')^{-2} dt', \quad \int_{-r_0}^t \mu_m^{-2b_{|\alpha|+2-n}+1}(t') (-t')^{-2} dt'$$

As in the previous sections, we split the interval $[-r_0, t]$ into two parts: $t' \in [-r_0, t_0]$ and $t' \in [t_0, t]$ where $\mu_m(t_0) = \frac{1}{10}$. If $t' \in [-r_0, t_0]$, we have

$$\begin{aligned} & \int_{-r_0}^{t_0} \mu_m^{-b_{|\alpha|+2-n}-1/2}(t') (-t')^{-2} dt' \lesssim \int_{-r_0}^{t_0} \mu_m^{-b_{|\alpha|+2-n}+1/2}(t') (-t')^{-2} dt' \lesssim \mu_m^{-b_{|\alpha|+2-n}+1/2}(t) \\ & \int_{-r_0}^{t_0} \mu_m^{-2b_{|\alpha|+2-n}+1}(t') (-t')^{-2} dt' \lesssim \int_{-r_0}^{t_0} \mu_m^{-2b_{|\alpha|+2-n}+2}(t') (-t')^{-2} dt' \lesssim \mu_m^{-2b_{|\alpha|+2-n}+2}(t). \end{aligned} \quad (10.15)$$

Here we used the fact that $\mu_m(t') \geq \frac{1}{10}$ for $t' \in [-r_0, t_0]$ and the second part of Lemma 7.1. For $t' \in [t_0, t]$, since

$$b_{|\alpha|+2-n} = [b_{|\alpha|+2-n}] + \frac{3}{4} \geq 1 + \frac{3}{4} = \frac{7}{4},$$

which implies

$$\eta_m - \frac{1}{2(b_{|\alpha|+2-n} - 1/2)} \geq c_0 > 0$$

for some absolute constant c_0 . Here η_m is defined by (7.9). Therefore the proof of Lemma 7.1 goes through and we have

$$\begin{aligned} \int_{t_0}^t \mu_m^{-b_{|\alpha|+2-n}-1/2}(t')(-t')^{-2} dt' &\lesssim \mu_m^{-b_{|\alpha|+2-n}+1/2}(t) \\ \int_{t_0}^t \mu_m^{-2b_{|\alpha|+2-n}+1}(t')(-t')^{-2} dt' &\lesssim \mu_m^{-2b_{|\alpha|+2-n}+2}(t). \end{aligned} \quad (10.16)$$

So indeed, we can repeat the process of 0th for $n = 1, \dots, [b_{|\alpha|+2}] - 1$. Therefore we have the following estimates:

$$\begin{aligned} \tilde{E}_{\leq |\alpha|+1-n}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+1-n}(t, \underline{u}) &\lesssim \mathcal{D}_{|\alpha|+2}^{\underline{u}} \\ \tilde{E}_{\leq |\alpha|+1-n}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+1-n}(t, \underline{u}) + \tilde{K}_{\leq |\alpha|+1-n}(t, \underline{u}) &\lesssim \delta \mathcal{D}_{|\alpha|+2}^{\underline{u}}. \end{aligned} \quad (10.17)$$

We now make the final step $n = [b_{|\alpha|+2-n}]$. In this case we have $b_{|\alpha|+2-n} = \frac{3}{4}$. Using the same process as in (10.2) and (10.5), the optical terms are bounded by:

$$\begin{aligned} \|Z^{\alpha'} \text{tr} \underline{X}'\|_{L^2(\Sigma_t^{\underline{u}})} &\lesssim \delta \mu_m^{-1/4}(t)(-t)^{-1} \sqrt{\mathcal{D}_{|\alpha|+2}^{\underline{u}}} \quad \text{with } |\alpha'| + 2 \leq |\alpha| + 1 - [b_{|\alpha|+2}] \\ \|Z_i^{\alpha'} T^{l'} \underline{\Delta} \mu\|_{L^2(\Sigma_t^{\underline{u}})} &\lesssim \delta^{1/2} \mu_m^{-1/4}(t)(-t)^{-1} \sqrt{\mathcal{D}_{|\alpha|+2}^{\underline{u}}} \quad \text{with } |\alpha'| + l' + 2 \leq |\alpha| + 1 - [b_{|\alpha|+2}] \end{aligned} \quad (10.18)$$

with $Z_i = R_i$ or Q . As before, in order to bound the corresponding integrals:

$$\int_{-r_0}^t \|Z_i^{\alpha'} \text{tr} \underline{X}'\|_{L^2(\Sigma_{t'}^{\underline{u}})}^2 dt', \quad \int_{-r_0}^t \|Z_i^{\alpha'} T^{l'} \underline{\Delta} \mu\|_{L^2(\Sigma_{t'}^{\underline{u}})}^2 dt'$$

we need to consider the integral:

$$\int_{-r_0}^t \mu_m^{-1/2}(t')(-t')^{-2} dt' \leq \int_{-r_0}^{t_0} \mu_m^{-1/2}(t')(-t')^{-2} dt' + \int_{t_0}^t \mu_m^{-1/2}(t')(-t')^{-2} dt' \quad \text{with } \mu_m(t_0) = \frac{1}{10}$$

For the “non-shock part $\int_{-r_0}^{t_0}$ ”, since $\mu_m(t_0) \geq \frac{1}{10}$,

$$\int_{-r_0}^{t_0} \mu_m^{-1/2}(t')(-t')^{-2} dt' \lesssim 1.$$

Let s be such that $t_0 \leq t' \leq t < s < t^*$. Following the same arguments as in (7.17) and (7.18), we have

$$\mu_m(t) \geq \mu_m(s) + (\eta_m - O(\delta M^4)) \left(\frac{1}{t} - \frac{1}{s} \right), \quad (10.19)$$

and

$$\mu_m(t) \leq \mu_m(s) + (\eta_m + O(\delta M^4)) \left(\frac{1}{t} - \frac{1}{s} \right). \quad (10.20)$$

If δ is sufficiently small, a similar argument as deriving (7.19) implies

$$\int_{t_0}^t \mu_m^{-1/2}(t') (-t')^{-2} dt' \lesssim \mu_m^{1/2}(t) \lesssim 1. \quad (10.21)$$

So we have the following bounds:

$$\begin{aligned} \left(\int_{-r_0}^t \|Z^{\alpha'} \text{tr} \underline{\chi}'\|_{L^2(\Sigma_{t'}^{\underline{u}})}^2 dt' \right)^{1/2} &\lesssim \delta \sqrt{\mathcal{D}_{|\alpha|+2}^{\underline{u}}} \quad \text{with} \quad |\alpha'| + 2 \leq |\alpha| + 1 - [b_{|\alpha|+2}], \\ \left(\int_{-r_0}^t \|Z_i^{\alpha'} T^{l'} \underline{\Delta} \mu\|_{L^2(\Sigma_{t'}^{\underline{u}})}^2 dt' \right)^{1/2} &\lesssim \delta^{1/2} \sqrt{\mathcal{D}_{|\alpha|+2}^{\underline{u}}} \quad \text{with} \quad |\alpha'| + l' + 2 \leq |\alpha| + 1 - [b_{|\alpha|+2}]. \end{aligned}$$

Therefore we can set:

$$b_{|\alpha|+1-n} = b_{|\alpha|+1-[b_{|\alpha|+2}]} = 0$$

in this step. Then we can proceed exactly the same as in the preceding steps. We thus arrive at the estimates:

$$\begin{aligned} \tilde{E}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) + \tilde{F}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) &\lesssim \mathcal{D}_{|\alpha|+2}^{\underline{u}} \\ \tilde{\underline{E}}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) + \tilde{\underline{F}}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) + \tilde{K}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) &\lesssim \delta \mathcal{D}_{|\alpha|+2}^{\underline{u}}. \end{aligned} \quad (10.22)$$

These are the desired estimates, because from the definitions:

$$\begin{aligned} \tilde{\underline{E}}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) &:= \sup_{t' \in [-r_0, t]} \{ \underline{E}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t', \underline{u}) \} \\ \tilde{E}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) &:= \sup_{t' \in [-r_0, t]} \{ E_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t', \underline{u}) \} \\ \tilde{\underline{F}}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) &:= \sup_{t' \in [-r_0, t]} \{ \underline{F}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t', \underline{u}) \} \\ \tilde{F}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) &:= \sup_{t' \in [-r_0, t]} \{ F_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t', \underline{u}) \} \\ \tilde{K}_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) &:= \sup_{t' \in [-r_0, t]} \{ K_{\leq |\alpha|+1-[b_{|\alpha|+2}]}(t', \underline{u}) \} \end{aligned} \quad (10.23)$$

the weight $\mu_m(t')$ has been eliminated.

11. COMPLETION OF PROOF

11.1. Proof of Theorem 3.1. Let us define:

$$\mathcal{S}_2[\phi] := \int_{S_{t, \underline{u}}} \left(|\phi|^2 + |R_{i_1} \phi|^2 + |R_{i_1} R_{i_2} \phi|^2 \right) d\mu_{\mathcal{G}}$$

And also let us denote by $\mathcal{S}_n(t, \underline{u})$ the integral on $S_{t, \underline{u}}$ (with respect to $d\mu_{\mathcal{G}}$) of the sum of the square of all the variation $\psi = \delta^{l'} Z_i^{\alpha'} \psi_{\gamma}$ up to order $|\alpha| + 1 - [b_{|\alpha|+2}]$, where l' is the number of T 's in the string of $Z_i^{\alpha'}$ and $\gamma = 0, 1, 2, 3$. Then by (5.16) we have:

$$\mathcal{S}_{|\alpha|-[b_{|\alpha|+2}]}(t, \underline{u}) \lesssim \delta \left(E_{|\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) + \underline{E}_{|\alpha|+1-[b_{|\alpha|+2}]}(t, \underline{u}) \right) \quad \text{for all} \quad (t, \underline{u}) \in [-r_0, t^*) \times [0, \delta].$$

Hence, in view of (10.22) and (10.23),

$$\mathcal{S}_{|\alpha|-[b_{|\alpha|+2}]}(t, \underline{u}) \lesssim \delta \mathcal{D}_{|\alpha|+2}^{\underline{u}} \quad \text{for all} \quad (t, \underline{u}) \in [-r_0, t^*) \times [0, \delta]. \quad (11.1)$$

Then for any variations ψ of order up to $|\alpha| - 2 - [b_{|\alpha|+2}]$ we have:

$$\mathcal{S}_2[\psi] \leq \mathcal{S}_{|\alpha|-[b_{|\alpha|+2}]}(t, \underline{u}) \quad (11.2)$$

Then by the isoperimetric inequality in (3.41), (11.1) and (11.2), we have:

$$\delta^{l'} \sup_{S_{t, \underline{u}}} |Z_i^{\alpha'} \psi_\alpha| = \sup_{S_{t, \underline{u}}} |\psi| \lesssim \delta^{1/2} \sqrt{\mathcal{D}_{|\alpha|+2}^{\underline{u}}} \leq C_0 \delta^{1/2} \quad (11.3)$$

where C_0 depends on the initial energy $\mathcal{D}_{|\alpha|+2}^{\underline{u}}$, the constant in the isoperimetric inequality and the constant in (5.16) as well as the constants in (10.22), which are absolute constants. If we choose $|\alpha|$ large enough such that

$$\left\lceil \frac{|\alpha| + 1}{2} \right\rceil + 3 \leq |\alpha| - 2 - [b_{|\alpha|+2}]$$

then (11.3) recovers the bootstrap assumption (B.1) for $(t, \underline{u}) \in [-r_0, t^*] \times [0, \delta]$.

To complete the proof of Theorem 3.1, it remains to show that the smooth solution exists for $t \in [-r_0, s^*)$, i.e. $t^* = s^*$. More precisely, we will prove that either $\mu_m(t^*) = 0$ if shock forms before $t = -1$ or otherwise $t^* = -1$.

If $t^* < s^*$, then μ would be positive on $\Sigma_{t^*}^\delta$. In particular μ has a positive lower bound on $\Sigma_{t^*}^\delta$. Therefore by Remark 2.4, the Jacobian Δ of the transformation from optical coordinates to rectangular coordinates has a positive lower bound on $\Sigma_{t^*}^\delta$. This implies that the inverse transformation from rectangular coordinates to optical coordinates is regular. On the other hand, in the course of recovering bootstrap assumption we have proved that all the derivatives of the first order variations ψ_α extend smoothly in optical coordinates to $\Sigma_{t^*}^\delta$. Since the inverse transformation is regular, ψ_α also extend smoothly to $\Sigma_{t^*}^\delta$ in rectangular coordinates. Once ψ_α extend to functions of rectangular coordinates on $\Sigma_{t^*}^\delta$ belonging to some Sobolev space H^3 , then the standard local existence theorem (which is stated and proved in rectangular coordinates) applies and we obtain an extension of the solution to a development containing an extension of all null hypersurface $\underline{C}_{\underline{u}}$ for $\underline{u} \in [0, \delta]$, up to a value t_1 of t for some $t_1 > t^*$, which contradicts with the definition of t^* and therefore $t^* = s^*$. This completes the proof of Theorem 3.1.

11.2. Data leading to shock formation. Finally, let us identify a class of initial data constructed in Lemma 1.1 which guarantee shock formation. As we have seen in Remark 2.4, to let shock form before $t = -1$, we need to let μ vanish before $t = -1$. By Proposition 4.5 we have

$$\mu(t, \underline{u}, \theta) = \mu(-r_0, \underline{u}, \theta) - \left(\frac{1}{t} + \frac{1}{r_0} \right) r_0^2 \underline{L}\mu(-r_0, \underline{u}, \theta) + O\left(\frac{\delta}{t^2}\right). \quad (11.4)$$

Note that since we already recovered all the bootstrap assumptions, the large parameter M in Proposition 4.5 goes away here. In view of the fact $\mu(-r_0, \underline{u}, \theta) = 1 + O\left(\frac{\delta}{r_0^2}\right)$, the propagation equation for μ as well as the pointwise estimates for ψ and its derivatives, we rewrite (11.4) as

$$\mu(t, \underline{u}, \theta) = 1 + 3r_0^2 \left(\frac{1}{|t|} - \frac{1}{r_0} \right) \left(G''(0)\phi_1 \left(\frac{r-r_0}{\delta}, \underline{u}, \theta \right) \partial_s \phi_1 \left(\frac{r-r_0}{\delta}, \underline{u}, \theta \right) \right) + O\left(\frac{\delta}{t^2}\right). \quad (11.5)$$

Here ∂_s is the partial derivative with respect to the first argument of the function $\phi_1(s, \theta)$. Therefore if (1.14) holds, $\mu(t, \underline{u}, \theta)$ becomes zero before $t = -1$. This completes the proof of the main theorem of the paper.

ACKNOWLEDGMENT

This work was supported by NSF grant DMS-1253149 to The University of Michigan and in its initial phase by ERC Advanced Grant 246574 ‘‘Partial Differential Equations of Classical Physics’’.

REFERENCES

- [1] Serge Alinhac. Temps de vie des solutions régulières des équations d'Euler compressibles axisymétriques en dimension deux. *Invent. Math.*, 111(3):627–670, 1993.
- [2] Serge Alinhac. Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions. II. *Acta Math.*, 182(1):1–23, 1999.
- [3] Serge Alinhac. Blowup of small data solutions for a quasilinear wave equation in two space dimensions. *Ann. of Math. (2)*, 149(1):97–127, 1999.
- [4] Demetrios Christodoulou. *The action principle and partial differential equations*, volume 146 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000.
- [5] Demetrios Christodoulou. *The formation of shocks in 3-dimensional fluids*. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2007.
- [6] Demetrios Christodoulou. *The formation of black holes in general relativity*. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2009.
- [7] Demetrios Christodoulou and Sergiu Klainerman. *The global nonlinear stability of the Minkowski space*, volume 41 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.
- [8] Demetrios Christodoulou and Shuang Miao. *Compressible flow and Euler's equations*, volume 9 of *Surveys of Modern Mathematics*. International Press, Somerville, MA; Higher Education Press, Beijing, 2014.
- [9] Demetrios Christodoulou and Daniel Raoul Perez. On the formation of shocks of electromagnetic plane waves in non-linear crystals. *J. Math. Phys.*, 57(8):081506, 56, 2016.
- [10] Gustav Holzegel, Sergiu Klainerman, Jared Speck, and Willie Wai-Yeung Wong. Small-data shock formation in solutions to 3D quasilinear wave equations: An overview. *J. Hyperbolic Differ. Equ.*, 13(1):1–105, 2016.
- [11] Fritz John. Formation of singularities in one-dimensional nonlinear wave propagation. *Comm. Pure Appl. Math.*, 27:377–405, 1974.
- [12] Fritz John. Blow-up of radial solutions of $u_{tt} = c^2(u_t)\Delta u$ in three space dimensions. *Mat. Apl. Comput.*, 4(1):3–18, 1985.
- [13] Fritz John. *Nonlinear wave equations, formation of singularities*, volume 2 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1990. Seventh Annual Pitcher Lectures delivered at Lehigh University, Bethlehem, Pennsylvania, April 1989.
- [14] Sergiu Klainerman. Global existence for nonlinear wave equations. *Comm. Pure Appl. Math.*, 33(1):43–101, 1980.
- [15] Sergiu Klainerman and Igor Rodnianski. On the formation of trapped surfaces. *Acta Math.*, 208(2):211–333, 2012.
- [16] Lev. D. Landau and Evgeny. M. Lifshitz. *Course of theoretical physics. Vol. 8*. Pergamon International Library of Science, Technology, Engineering and Social Studies. Pergamon Press, Oxford, 1984. Electrodynamics of continuous media, Translated from the second Russian edition by J. B. Sykes, J. S. Bell and M. J. Kearsley, Second Russian edition revised by Lifshits and L. P. Pitaevskii.
- [17] Jonathan. Luk and Jared. Speck. Shock formation in solutions to the 2D compressible Euler equations in the presence of non-zero vorticity. *ArXiv e-prints*, October 2016.
- [18] Shuang. Miao and Pin. Yu. On the formation of shocks for quasilinear wave equations. *To appear on Invent. Math.*, 2016.
- [19] Jared. Speck. Shock Formation in Small-Data Solutions to 3D Quasilinear Wave Equations. *ArXiv e-prints*, July 2014.
- [20] Jared. Speck, Gustav. Holzegel, Jonathan. Luk, and Willie Wai-Yeung. Wong. Stable shock formation for nearly plane symmetric waves. *ArXiv e-prints*, January 2016.